Distinguishing Factors and Characteristics with Characteristic-Mimicking Portfolios *

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Abstract

We advance a procedure for deriving systematic factors from characteristics based on maximizing each factor's exposure to a characteristic for given factor variance. The resulting characteristic-mimicking portfolios (CMPs) price assets identically as the original characteristics and have maximum power to identify underlying factors. Performance differences of mimicking factors and characteristics in explaining mean returns are artifacts of arbitrary procedures for generating mimicking factors. CMPs are ideally suited to distinguish factors and characteristics by explanatory power for the time series of returns and are useful for improving risk management and to determine if return explanations are justifiably linked to systematic risk.

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Abstract

We advance a procedure for deriving systematic factors from characteristics based on maximizing each factor's exposure to a characteristic for given factor variance. The resulting characteristic-mimicking portfolios (CMPs) price assets identically as the original characteristics and have maximum power to identify underlying factors. Performance differences of mimicking factors and characteristics in explaining mean returns are artifacts of arbitrary procedures for generating mimicking factors. CMPs are ideally suited to distinguish factors and characteristics by explanatory power for the time series of returns and are useful for improving risk management and to determine if return explanations are justifiably linked to systematic risk.

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1. Introduction

A large literature has attempted to disentangle whether cross-sectional differences in asset returns are best explained by exposure to systematic factor risk or by asset-specific characteristics. Our paper argues that, for the explanation of mean returns, the distinction between risk factors and characteristics has no empirical meaning, and that advantages of using one approach over the other are mostly of a technical nature. However, for the explanation of variations in returns, it is possible to make a meaningful distinction between factors and characteristics. Given a natural choice of a characteristic-mimicking portfolio (CMP) as a factor, the covariance (or the factor loading) of each asset with the factor is proportional to the asset's characteristic. It follows that using either the CMP or the characteristic to explain mean returns provides identical results. But the CMP varies substantially across time and, potentially, can explain a significant degree of time series variation in returns. If so, the CMP has value for hedging purposes and the factor formulation may then dominate the characteristics formulation.

Following the influential work of Fama and French (1993, 1996) it has become commonplace to explain anomalies in asset pricing with covariance-risk factors constructed as portfolios that are long on securities of firms with high values of a particular characteristic and short on securities of firms with low values of the characteristic. In this formulation, the erstwhile anomalies may be reinterpreted as rewards for risk associated with the constructed risk factor.

Starting with Daniel and Titman (1997), Jagannathan and Wang (1996), and Daniel, Hirshleifer, and Subrahmanyam (2001), a substantial literature has attempted to differentiate between covariance-risk and characteristics-based explanations for asset returns, with varying results.[1] Ferson, Sarkissian, and Simin (1999) question whether it is possible to effectively distinguish between risk factors and characteristics. They employ a variant of the Fama and French approach, first suggested by Fama (1976, pp. 326-329) who pointed out that estimates of risk premia based on so-called Fama-MacBeth (1973) regressions, potentially using characteristics, may be interpreted as portfolios of the assets, with the portfolio weights depending on the characteristics. The risk-premia here in effect may be viewed as factors that mimic characteristics. Using this approach, Ferson et al. (1999) find that a nonsense factor such as the alphabet factor they create

[1] Recent papers include Hou, Karolyi and Kho (2011), Daniel and Titman (2012), Chordia, Goyal, and Shanken (2015), Luo and Balvers (2017), and Pukthuanthong and Roll (2014).

based on company names becomes a significant risk factor. In addition to showing that it is easy to generate a risk factor, they show at the same time that it is difficult to distinguish the effect of the characteristics from the effect of a risk factor mimicking this characteristic.

Balduzzi and Robotti (2008) take the argument a step further. Providing an econometric analysis of the difference between using a two-pass approach versus using factor-mimicking portfolios to evaluate the performance of non-tradable factors, they additionally point out an observational equivalence between characteristics-based and risk-based models, showing that it is always possible to create a factor that implies factor loadings which for each asset are equal to the asset characteristic. While they do not further pursue the implications, they define the CMPs that we will be using here (with minor differences). The CMPs are also quite similar to the mimicking portfolios discussed by Fama (1976) and Ferson et al. (1999) except that the portfolios they consider do not imply that factor loadings and characteristics are aligned unless the covariance matrix of asset returns is diagonal.

A recent paper by Kozak, Nagel, and Santosh (2018) provides a quite different argument for why it is difficult to distinguish risk factors and characteristics explanations. If some investors are rational in the conventional sense and some invest with behavioral motives then the behavioral investors focusing on particular asset characteristics may cause price deviations, presenting opportunities for rational investors. If the deviations correlate with a systematic risk factor the "arbitrage" will not eliminate much; if the deviations are uncorrelated with systematic risk the arbitrage should eliminate most of the price deviations. In either case, however, the pricing in equilibrium is consistent with the first-order conditions of both types of investors. Focusing on the behavioral investor first-order conditions one would find a characteristics explanation; focusing on the rational investor first-order conditions a risk-based explanation works better. Yet both views explain the same price observations.

Our argument goes further in that attempting to distinguish risk-factor and characteristics explanations empirically is pointless to begin with for asset pricing purposes, i.e., explaining the mean returns of assets. To evaluate if a characteristic or a related risk factor works better, it is not reasonable to compare their performances in explaining cross-sectional differences in mean returns: either the performances are exactly identical or the characteristic-mimicking factor was obtained ad hoc so that any performance difference is arbitrary. Characteristics and factors, nevertheless, may still be usefully differentiated by their *explanatory power for return shocks*: while the risk premia for characteristics are viewed as constant over time, the risk premia for the CMP loadings are the factor realizations which must vary stochastically over time. If the CMPs explain a significant part of return variation they are valuable for hedging risk and portfolio management, and dominate the corresponding characteristics formulation. The same procedure for comparing factor and characteristic specifications works just as well if the original formulation is in the risk factor form, in which case we would simply use the loadings on this risk factor to serve as the characteristics.

We follow Pukthuanthong and Roll (2014) in considering whether a factor is truly a risk factor by evaluating both if the factor has significant pricing power for the assets *and* whether it explains a significant part of return variability. However, presupposing an arbitrage pricing context Pukthuanthong and Roll (2014), as do Kozak et al. (2018), argue that, insofar as characteristics do not match the loadings on systematic risk factors, priced characteristics represent near-arbitrage opportunities (very large Sharpe ratios). Thus, for them the identifying criterion of (priced) characteristics vis-à-vis factor loadings is that they represent potential for high Sharpe ratios. The drawback of using Sharpe ratios in this manner is that we know that the maximum Sharpe ratio is that of the (ex post) tangency portfolio and a higher Sharpe ratio means mechanically higher correlation with this tangency portfolio. Thus, translating their criterion to our terms, the CMP is a "good" factor if it has a high Sharpe ratio. But this does not rule out a characteristics interpretation; it merely suggests that a group of investors cares a lot about the characteristic for this interpretation to hold. So, it is not clear, definitely not from a practical perspective, what Sharpe ratio should be considered too high to rule out a characteristics explanation. On the other hand, in our approach we simply admit that both interpretations are equally sound as far as explaining mean returns is concerned. Subsequently, we check if the CMP explains a significant extra quantity of return variability. If so, interpreting the variable as a risk factor is fruitful because it allows for better risk management.

Concurrent research by Kelly, Pruitt, and Su (2018) provides a general approach that distinguishes characteristics from factor loadings. The approach relates alphas as well as betas on a predetermined set of latent factors linearly to a group of characteristics. The coefficients relating the characteristics to alphas and latent factor betas are estimated to minimize the sum of squared return errors over all assets and time periods, as in our case. Significant coefficients on the alpha part indicate characteristics effects while significant coefficients on the beta parts indicate systematic risk effects. The approach allows for time-variation in the characteristics and has much to recommend it. However, it differs in important ways from our approach. First, our CMP approach naturally imposes the prior understanding that loadings on factors should be closely related to the characteristics that generate apparent anomalies, if there is a risk-based explanation. Second, Kelly et al. (2018) view as true risk factors any possible portfolio that allows characteristics to combine with it to explain returns. Our CMP approach avoids this bias toward the risk-based explanation. Third, our CMP approach obtains simple closed-form solutions for the latent risk factors. Fourth, our CMP approach allows us to focus on one characteristic at a time.

In addition to avoiding focus on non-material differences among factor proxies, CMPs are particularly well suited for testing an underlying factor's explanatory power for the time-series of asset returns. A property of the CMP as the factor having highest possible exposure to the characteristic for given variance implies that it has high power for discerning significant explanatory power, and is ideally suited for testing if a variable with significant explanatory power for mean returns should be viewed as a risk factor or as a characteristic. The result not only improves the usage of the variable for risk management purposes, but also establishes a way for deciding if returns are best viewed as compensation for risk or not, in which case they are determined by behavioral aspects or, more generally, by preferences for non-pecuniary attributes of an asset.

In the following, we first discuss further the literature in Section 2 and then discuss the properties of CMPs in Section 3. We present in Section 4 a direct comparison of characteristics and risk factor formulations and discuss in Section 5 how they may be distinguished theoretically. In Section 6 we present simulations that illustrate the power of using CMPs to identify variables that are risk factors, in comparison to the power of Factor Mimicking Portfolios (FMPs) of the type adopted by Fama and French). Section 7 provides empirical results of distinguishing the characteristics and factor formulations for a standard set of characteristics, employing industry portfolios and factor portfolios as test assets. Section 8 concludes.

2. Context

Fama and French (1992) established empirically that the size (average log of market value) characteristic and value (average log of book-to-market ratio) characteristic of stock portfolios explained differences in mean returns across portfolios much better than market factor loadings. Fama and French (1993) then constructed risk factors based on the size and value characteristics and concluded that these mimicking portfolios functioning as risk factors explained average asset returns in accordance with equilibrium pricing models. Their construction of the mimicking portfolios lacked a formal motivation but consisted roughly of taking the return of the smallest 50% of firms minus the return of the biggest 50% of firms in each period as the size-mimicking factor return; and taking the return of the 30% highest book-to-market value firms minus the return of the 30% lowest book-to-market value firms as the value-mimicking factor return.

As both size and value characteristics and size and value risk factors separately performed well in explaining average return differences, the natural question became which one performed better. Daniel and Titman (1997) and Jagannathan and Wang (1996) introduced simple approaches for comparing the importance of characteristics and the mimicking risk factors derived from the characteristics. Daniel and Titman sorted portfolios separately by factor loadings and by characteristics and compared the return differences of the sorted portfolios. Jagannathan and Wang added both the factor loadings and the characteristics themselves to a regression explaining cross-sectional differences in mean returns. The regression results determined whether factor loadings drove out the characteristics or vice versa.

Extensive and continuing application of both approaches has led to diverging results, with sometimes characteristics beating factor loadings (e.g., Brennan, Chordia, and Subrahmanyam 1998, and Chordia, Goyal, and Shanken 2015), at other times factor loadings beating characteristics (e.g., Davis, Fama, and French 2000, and Gao 2011), or the results varying by characteristic (Hou et al. 2011). Daniel and Titman (2012) argue that a sharper empirical distinction is needed in creating separate portfolios based on characteristics and factor loadings to accurately identify which works better.

Recent literature has attempted to further clarify the distinction between characteristics and factors. Lin and Zhang (2011) appeal to the production-based asset pricing context in which firm characteristics are related to investment returns and thus naturally represent loadings on investment risk which must relate to return risk. Presupposing an arbitrage pricing context Kozak et al. (2018) and Pukthuanthong and Roll (2014) argue that, insofar as characteristics do not match the loadings on systematic risk factors, priced characteristics represent near-arbitrage opportunities (very large Sharpe ratios). Thus, the identifying criterion of (priced) characteristics vis-à-vis factor loadings is that they represent potential for high Sharpe ratios. Kozak et al. (2018) further provide a theoretical model that illustrates that it is not possible to distinguish irrational from rational explanations for market outcomes. They argue that investors irrationally focusing on asset characteristics may cause price deviations, presenting opportunities for rational investors. If the deviations correlate with a systematic risk factor the "arbitrage" will not eliminate much; if the deviations are uncorrelated with systematic risk the arbitrage should eliminate most of the price deviations.

Gao (2011) provides an approach for employing characteristics to model asset covariances based on the similarity of assets in terms of their characteristics. The covariances now perform better than factor loadings in explaining return differences across assets and drive out the characteristics. This supports the view that it is risk factors, although not approximated through factor loadings, which are more relevant than behavioral factors proxied by characteristics. Moskowitz (2003), Connor and Linton (2007) and Suh, Song, and Lee (2014) also provide methods for relating factor loadings (or, similarly, covariances) to characteristics that differ from the approach of Fama and French (1993). Taylor and Verrecchia (2015) show that given delegation of investment both risk factors and individual characteristics will be priced. Chordia, Goyal, and Shanken (2015) contribute to the debate on loadings versus characteristics by focusing on individual stocks rather than portfolios and adjusting for the substantial measurement error bias that results from estimating loadings for individual stocks. They find that characteristics perform relatively better than factor loadings in explaining average return differences.

The intent in the recent literature is to sharpen the distinction between characteristics and factor loadings, whereas our objective in part is the opposite: to emphasize that the distinction between characteristics and factor loadings is immaterial for explanations of mean returns. Characteristics are identical to factor loadings but on a factor that is generally only trivially different (in construction, not by impact as Kogan and Tian, 2015, argue). The choice of CMP is dictated objectively by a procedure that maximizes exposure of the factor portfolio to the underlying characteristic subject to a particular level for the return variance of the mimicking factor. But the factor mimicking portfolio based on the Fama and French approach is chosen haphazardly, so the difference with the characteristic mimicking portfolio is not fundamental, and basing tests on the difference between the two seems beside the point.

Consider the iconic example of the size characteristic. The Fama-French approach (henceforth *FF approach*) derives from the idea of Fama and French (1993, 1996) to explain the empirical importance of the size characteristic for average returns from a risk-taking perspective: for whatever reason, smaller-size firms are more exposed to systematic risk. Accordingly, Fama and French construct a risk factor as the return of a factor-mimicking portfolio formed, roughly, by holding the 50% smallest firms and shorting the 50% largest firms. Empirically, the resulting "size factor" helps to explains differences in mean returns well.

However, why construct the risk factor in this manner? A theoretical motivation for the construction would protect against data mining through specification search. If the idea is that smaller firms are more sensitive to risk, why not construct a risk factor such that the sensitivity to this factor is indeed directly related to firm size? The latter defines the approach advocated in the present paper (henceforth the *CMP approach*): construct a risk factor such that the covariance (or the factor loading) of each asset's return with this factor is identical to the characteristic of each asset. This Characteristic-Mimicking Portfolio (CMP) also provides the theoretical justification that it is the minimum variance portfolio with the largest exposure to the characteristic (firm size in this example) and it provides the maximum degree of information about the latent risk factor.

It is not our intent to provide a factor that is simply a variant of the size factor generated by Fama and French. Our key point is that comparing factor loadings and characteristics is mostly meaningless. Trivially the tests initially proposed by Daniel and Titman and Jagannathan and Wang (see also Jagannathan, Skoulakis, and Wang, 2010) cannot be performed when CMPs are the mimicking portfolios. While other mimicking portfolios, typically generated from ad hoc assumptions, may produce differences between factor loadings and characteristics, these differences are nonessential from a theoretical perspective, even though Kogan and Tian (2015) argue they may nevertheless be essential empirically.

Mimicking portfolios were first advocated to represent *macro risk factors* by Breeden (1979), Grinblatt and Titman (1987), and Huberman, Kandel, and Stambaugh (1987) and applied to represent consumption risk by Breeden, Gibbons, and Litzenberger (1989). These mimicking portfolios convert systematic risk tied to realizations of macro-economic variables into tradable asset portfolios with returns that explain average asset returns just as well as the original macro factors. Lamont (2001) devised an alternative construction of mimicking portfolios, "tracking portfolios", to represent *expectations of macro variables*. As a key application, Kapadia (2011) used this approach to capture distress risk. Ferson, Spiegel, and Xu (2006) consider the optimal use of mimicking portfolios representing macro risk factors in the context of predictable time variation.

Much earlier, Fama (1976, pp. 326-329) pointed out (see also Ferson et al., 1999) that estimates of risk premia based on Fama-MacBeth (1973) regressions, potentially using characteristics, may be interpreted as portfolios of the assets, with the portfolio weights depending on the characteristics. Here the mimicking portfolios are used to represent *individual asset characteristics*. The CMP version arising in Balduzzi and Robotti (2008), although not previously applied for this purpose, precisely mimics characteristics in the sense that assets' loadings on such a mimicking portfolio directly reproduce the assets' characteristics. The approach of Fama and French (1993) provides an alternative, although without formal validation, also generating mimicking portfolios interpreted as risk factors that represent characteristics. Back, Kapadia and Ostdiek (2013) empirically consider the performance of both the Fama (1976) and the Fama and French (1993) approaches, terming aptly the Fama (1976) mimicking portfolios "characteristic pure plays." Kirby (2018) adapts the Fama (1976) approach to enable sorting based on unexplained characteristics components and constructing mimicking portfolios more efficiently. [2]

[2] Not only does the CMP approach mimic an aggregate risk associated with a desired characteristic, it also provides a practical vehicle for estimating firm-level characteristics based on measuring factor loadings on the mimicking portfolio, as we explore separately in a follow-up paper: By constructing a mimicking portfolio for a specific characteristic it becomes possible to estimate a future value of a firm's characteristic (before it is observed) from observation of the firm's factor loading on the CMP, which is typically observed at a higher frequency than the characteristic itself. As such, using mimicking portfolios is akin to Lamont (2001) in the limited sense that it can be used to provide estimates of unobservable variables. But rather than generating estimates of expectations of macro variables, we use the CMP to provide estimates of firm-level characteristics that cannot be observed in real time.

3. Characteristic-Mimicking Factors

It is always possible to create a "characteristic-mimicking portfolio" (CMP) to function as an additional factor that prices all assets in the same way as the original characteristics, and converts the premium associated with a deterministic set of asset characteristics to a premium for systematic risk associated with a risk factor that changes stochastically over time.

Following Balvers and Luo (2016), define a characteristic-mimicking factor as a portfolio of the risky assets that: (1) maximizes the exposure to the characteristic, subject to (2) a particular portfolio variance. The covariance matrix of the returns of N risky assets is given by a positive definite $\mathbf{\Sigma}$. A particular characteristic for all assets is represented by the vector $\mathbf{z}$; $\mathbf{s}_{\mathbf{z}}$ is the vector of portfolio shares of the characteristic-mimicking portfolio; and $\bar{\sigma}^2$ is the pre-determined portfolio variance.

$$\underset{\mathbf{s}_{\mathbf{z}}}{\text{Max}} \; (\mathbf{s}_{\mathbf{z}}'\mathbf{z}), \qquad \text{s. t.} \quad \mathbf{s}_{\mathbf{z}}'\mathbf{\Sigma}\mathbf{s}_{\mathbf{z}} = \bar{\sigma}^2 , \tag{1}$$

Given the Lagrangian formulation with multiplier ½ λ, the first-order conditions become

$$\mathbf{s}_{\mathbf{z}} = (1/\lambda)\, \mathbf{\Sigma}^{-1}\, \mathbf{z}, \tag{2}$$

which provides the portfolio shares of the zero-investment characteristic-mimicking factor with return $r_{\mathbf{z}} = (1/\lambda)\, \mathbf{r}'\, \mathbf{\Sigma}^{-1}\, \mathbf{z}$. Note that the scale as affected by λ is unimportant for the factor choice since we have a zero-investment portfolio and, even if we did not have a zero-investment portfolio, would have no impact on the explanatory power of the factor.

For empirical purposes we choose $\bar{\sigma}^2$ for each characteristic so that $\lambda = 1$ for each mimicking factor. In this case it follows that $r_{\mathbf{z}} = \mathbf{r}'\, \mathbf{\Sigma}^{-1}\, \mathbf{z}$ and $\mu_{\mathbf{z}} = \boldsymbol{\mu}'\, \mathbf{\Sigma}^{-1}\, \mathbf{z}$, where $\mathbf{r}$ and $\boldsymbol{\mu}$ are N x 1 vectors of returns and mean returns, respectively, so that:

$$Cov(r, r_z) = E[(\mathbf{r} - \boldsymbol{\mu})(r_{\mathbf{z}} - \mu_{\mathbf{z}})] = E[(\mathbf{r} - \boldsymbol{\mu})(\mathbf{r} - \boldsymbol{\mu})']\mathbf{\Sigma}^{-1}\, \mathbf{z} = \mathbf{z} \tag{3}$$

Thus, for any set of characteristics it is possible to create a CMP factor for which the covariance with any of the assets generates the asset's characteristic. In the next section we show that for pricing purposes one may replace a particular risk factor by a set of deterministic characteristics, or vice versa replace a set of deterministic characteristics by an equivalent systematic risk factor, without changing the pricing results.

Maximizing the signal about the underlying factor

A further property of CMPs stems from the fact that the characteristic, when viewed as an unbiased indicator of a loading on an underlying factor, captures the underlying factor optimally.

View the *N x* 1 vector of returns $\mathbf{r}_t$ at each time *t* as determined by a set of factors we do not need to specify where $\mathbf{r}_t^O$ is the *N x* 1 vector capturing each asset's sum of factor risk premia plus idiosyncratic risk; and focus on one additional latent factor *F* with return r_t^F and factor loadings $\mathbf{b}$ that, without loss of generality, may be considered to have zero mean and be uncorrelated with the other factors (otherwise we may just focus on the uncorrelated component as the latent factor).

$$\mathbf{r}_t = \mathbf{r}_t^O + \mathbf{b}\, r_t^F \ . \qquad (4)$$

The loadings are assumed to be imperfectly correlated with the *N x 1* vector of characteristics $\mathbf{z}$:

$$\mathbf{b} = \mathbf{z} + \boldsymbol{\omega} \ , \qquad (6)$$

where $\boldsymbol{\omega}$ is an *N x 1* vector of mean-zero random components. The covariance matrix of the returns is then given by

$$\boldsymbol{\Sigma} = \boldsymbol{\Sigma}^O + \sigma_F^2\, \mathbf{z}\mathbf{z}' + (\sigma_F^2 + \mu_F^2)\,\boldsymbol{\Omega}, \qquad (5)$$

with $\boldsymbol{\Sigma}$ the full covariance matrix of the returns; $\boldsymbol{\Sigma}^O$ the covariance matrix for the unspecified factors plus idiosyncratic risk; $\boldsymbol{\Omega}$ the covariance matrix of the $\boldsymbol{\omega}$; and μ_F and σ_F^2 the mean and the variance of the unobserved factor return.

Then we wish to design a mimicking factor $r_t^{MIM} = \mathbf{s}'\mathbf{r}_t$ for *F* that captures its fluctuations well.

$$r_t^{MIM} = \mathbf{s}'\mathbf{r}_t = \mathbf{s}'(\mathbf{r}_t^O + \boldsymbol{\omega}\, r_t^F) + \mathbf{s}'\,\mathbf{z}\, r_t^F \ . \qquad (7)$$

The first term on the right-hand side is the noise in the mimicking return that is unrelated to characteristics **z** and the second term is the signal. The criterion is that the mimicking factor maximizes the variation due to signal about the true factor relative to the total variation. Taking the variance in equation (7):

$$\mathbf{s}'\mathbf{\Sigma}\mathbf{s} = \mathbf{H} + \sigma_F^2 (\mathbf{s}'\mathbf{z})^2 . \qquad (8)$$

Here $\mathbf{H} \equiv \mathbf{s}'\mathbf{\Sigma^O}\mathbf{s} + (\sigma_F^2 + \mu_F^2)\mathbf{s}'\mathbf{\Omega}\mathbf{s}$ and does not depend on **z**. The relative signal content in the mimicking factor is given by signal variance to total variance ratio:

$$\text{Signal ratio} = \sigma_F^2 (\mathbf{s}'\mathbf{z})^2 / \mathbf{s}'\mathbf{\Sigma}\mathbf{s} \ . \qquad (9)$$

The ratio involves the cross-moment between factor shares and characteristics relative to the variance of the factor return. The discussion earlier in this section shows how to maximize this ratio: for any given variance $\mathbf{s}'\mathbf{\Sigma}\mathbf{s}$ the ratio is maximized by maximizing $\mathbf{s}'\mathbf{z}$. As shown in equation (2) the maximum of the ratio in equation (9) occurs for

$$\mathbf{s} = (1/\lambda)\, \mathbf{\Sigma}^{-1}\, \mathbf{z} . \qquad (10)$$

Using the portfolios shares from equation (10) the maximum signal ratio becomes $\sigma_F^2 (\mathbf{z}'\mathbf{\Sigma}^{-1}\mathbf{z})$. Higher factor return variance as well as larger characteristics exposures for each asset relative to what this asset contributes to variance of all returns enhance the ability to identify the factor.

Thus, the CMP has one further interpretation. Apart from generating covariances with the assets equal to the asset characteristics and maximizing exposure to the characteristic for given variance, it also maximizes the informative content about the underlying factor. The latter property is important since it implies that a mimicking factor of this type is particularly informative in explaining time series fluctuations in returns. E.g. in the time-series regressions

$$\mathbf{r}_t = \boldsymbol{\alpha} + \boldsymbol{\beta} r_t^{MIM} + \boldsymbol{\varepsilon}_t , \qquad (11)$$

The factor $r_t^{MIM} = \mathbf{s}'\mathbf{r}_t$ is highly informative about the underlying factor, even though its variance may be relatively low, and is expected to provide the clearest explanatory power of any mimicking portfolio. We will provide simulation results supporting that this is indeed the case.

4. Empirical Specification

We present next a general empirical formulation. To distinguish characteristics and risk factor loadings explanations we present two simple models (similar to the models in Balduzzi and Robotti, 2008). In the first a set of characteristics (in addition to regular systematic risk factors) linearly affects returns for all assets and time periods. In the second the characteristics are replaced with risk factors which also (partially) explain returns for all assets but with stochastic risk premia in all periods, as is necessary for a risk factor. We then generate the CMPs. Once we obtain the CMPs we discuss how they can be used to suitably distinguish characteristics and risk factor specifications.

We focus here on the development and application of CMPs which we may define outside of the context of a particular model and for cases in which asset prices are explained only partially. Consider a given set of *N* firms issuing financial assets. The firms are characterized by *K* different characteristics captured by the *N x K* matrix $\mathbf{Z}$, which we assume to be constant over time. Under the *characteristics view*, for each time period we have:

$$\mathbf{r}_t = \mathbf{Zc} + \mathbf{e}_t , \tag{12}$$

where $\mathbf{r}_t$ is a *N x 1* vector of excess returns with *N x 1* vector of time series means $\boldsymbol{\mu}$ of the asset excess returns and $\boldsymbol{\Sigma}$ the covariance matrix of the asset excess returns, that may already have been adjusted for known factor risk. $\mathbf{c}$ is *K x 1*, and $\mathbf{e}_t$ is the *N x 1* vector of errors. Note that $\mathbf{Z}$ may include $\mathbf{1}_N$, a *N x 1* vector of ones, to capture a constant in the characteristics estimation (we would then define $\mathbf{Z} = \begin{pmatrix}\mathbf{1_N} & \mathbf{X}\end{pmatrix}$, so that $\mathbf{X}$ represents a *N x (K-1)* matrix of non-constant characteristics, and $\mathbf{c} = \begin{pmatrix} a & \mathbf{b}\end{pmatrix}'$). Pool over all time periods to estimate $\mathbf{c}$. If we estimate $\mathbf{c}$ efficiently from eq. (12) by Generalized Least Squares (GLS) then we obtain

$$\mathbf{c} = (\mathbf{Z}'\boldsymbol{\Sigma}^{-1}\mathbf{Z})^{-1}\mathbf{Z}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu} , \tag{13}$$

Alternatively, we have the *risk factor specification* with tradable assets serving as risk factors

$$\mathbf{r_t} = \boldsymbol{\alpha} + \mathbf{B}\mathbf{r_t^Z} + \mathbf{u_t} , \tag{14}$$

Where $\mathbf{r}_t$ is as above, $\boldsymbol{\alpha}$ is *N x 1*. $\mathbf{u_t}$ is i.i.d. *N x 1* with means equal to zero. $\mathbf{r_t^Z} = \mathbf{S'r_t}$ which is *K x 1* and $\mathbf{S}$ is an *N x K* matrix of portfolio shares. Suppose the *K* risk factors are chosen according to equation (10), with $\lambda = 1$ and aggregating all characteristics, to be the following portfolios of the *N* assets:

$$\mathbf{S} = \mathbf{\Sigma}^{-1}\mathbf{Z}, \tag{15}$$

where $\mathbf{\Sigma} = \mathbf{E[(r_t - \boldsymbol{\mu}_t)(r_t - \boldsymbol{\mu}_t)']}$ is the covariance matrix of the *N* asset returns. Thus, $Cov(\mathbf{r_t}, \mathbf{r_t^Z}) = E[(\mathbf{r_t} - \boldsymbol{\mu})(\mathbf{r_t} - \boldsymbol{\mu})']\mathbf{\Sigma}^{-1}\mathbf{Z} = \mathbf{Z}$.

Given eq. (14) we find the "betas" from the standard first-pass time series regressions:

$$\mathbf{B} = \mathbf{\Sigma S(S' \Sigma S)}^{-1}. \tag{16}$$

From eqs. (15) and (16) we may infer that $\mathbf{B} = \mathbf{Z(Z' \Sigma^{-1} Z)}^{-1}$. Taking expectations in both models (12) and (14) implies that $\boldsymbol{\alpha} = E(\mathbf{e}_t)$, where the expectation represents the time series average. Both models provide the same estimates for mean returns. It follows that we can generate a CMP for every set of assets and every characteristic.

If $\mathbf{Z}$ includes a unit vector to add a constant in the specification, so that $\mathbf{Z} = (\mathbf{1_N} \quad \mathbf{X})$, then the set of mimicking portfolios must be supplemented with a constant-mimicking portfolio which has investment weights $\mathbf{S_C} = \mathbf{\Sigma}^{-1}\mathbf{1_N}$ and with which every asset has identical unit covariance. This portfolio is up to scaling equal to the global minimum variance portfolio and has zero explanatory power for differences in mean return across the assets. The lack of cross-sectional explanatory power is intuitively clear because all assets load equally on this factor.

If $\mathbf{Z}$ consists of a vector $\boldsymbol{\mu}$ of the mean returns of the test assets then the mimicking portfolio is (again up to a scaling factor) equal to the tangency portfolio: $\mathbf{S_T} = \mathbf{\Sigma}^{-1}\boldsymbol{\mu}$. By design, therefore, the "mean return" characteristics as well as the associated CMP perfectly explain the mean returns of all test assets. Arbitrage Pricing Theory implies that a factor explaining all of the mean returns should also explain all of the undiversifiable risk. Accordingly, the CMP for mean returns (equal to the tangency portfolio) also should explain more of the time series variation than any other factor. Note that this implication must hold for arbitrage pricing theories but not necessarily for equilibrium asset pricing theories.

The two models discussed assume that characteristics and factor loadings are constant over time. This captures the idea that, by nature, factor loadings and characteristics change very slowly compared to factor realizations which have essentially no persistence. It is possible to allow characteristics (or loadings) to change dramatically every period so that the characteristics model could explain time-series fluctuations as well. However, such a formulation is counter to the intuitive notion of a characteristic as being a relatively stable attribute of an asset. On the other hand, allowing characteristics to evolve slowly over time can easily be accommodated in our approach. It would involve using actual characteristics at each time instead of averages over time and estimating betas as rolling averages (using the standard 60 periods, for instance) rather than using all previous data points. As we believe the empirical impact of this generalization to be minor we have not incorporated the time variation in characteristics and loadings in our approach and in the simulations and estimation to be discussed.

5. Distinguishing Factors and Characteristics

With properly chosen CMP there is no difference between characteristics and covariance with the factor, in terms of pricing (i.e. explaining average returns). However, the characteristics have a constant impact on returns whereas the risk factor realizations are stochastic in each time period. As a result, the factors may explain *variation* in realized returns better. Comparing the unexpected returns for both models:

$$\mathbf{r_t} - \boldsymbol{\mu} = \mathbf{e}_t - E(\mathbf{e}_t)\,. \qquad (12')$$

$$\mathbf{r_t} - \boldsymbol{\mu} = \mathbf{B}(\mathbf{r_t^Z} - \boldsymbol{\mu}^{\mathbf{Z}}) + \mathbf{u_t}\,, \qquad (14')$$

where $\boldsymbol{\mu}^{\mathbf{Z}} = \mathbf{S}'\boldsymbol{\mu}$. The right-hand sides are equal in both specifications, and both represent unpredictable components, but the error in eq. (14') includes the *factor risk component* which may be hedged to eliminate part of the risk. So, for the factor model to be better it must be that the variance of error $\mathbf{u_t}$ in eq. (14') is significantly less than the variance of error $\mathbf{e}_t - E(\mathbf{e}_t)$ in eq. (12'). In that case, a hedging strategy of holding asset *i* and shorting $\mathbf{Z}_i(\mathbf{Z}'\boldsymbol{\Sigma}^{-1}\mathbf{Z}\ )^{-1}$ units of each factor should significantly reduce risk ($\mathbf{Z}_i$ is the row vector of asset i's K characteristics), which could be true only by coincidence under the characteristics interpretation.

We use as the performance criterion the equal-weighted average of the error variance pooled over all time periods and assets. The reduction in error variance (equal-weighted) due to **Z** alone then equals $Tr\{E[\mathbf{B}(\mathbf{r_t^Z}-\boldsymbol{\mu}^\mathbf{Z})(\mathbf{r_t^Z}-\boldsymbol{\mu}^\mathbf{Z})'\mathbf{B}']\}$, where Tr represents the trace of the matrix. Given $\boldsymbol{\mu}^\mathbf{Z}=\mathbf{S}'\boldsymbol{\mu}$ and $\mathbf{S}=\boldsymbol{\Sigma}^{-1}\mathbf{Z}$, then, using eq. (16),

$$Tr(\boldsymbol{\Sigma})-Tr(\mathbf{uu'})=Tr[(\mathbf{e}_t-E(\mathbf{e}_t))(\mathbf{e}_t-E(\mathbf{e}_t))']-Tr(\mathbf{uu'})=Tr[\mathbf{Z}(\mathbf{Z}'\boldsymbol{\Sigma}^{-1}\mathbf{Z})^{-1}\mathbf{Z}'], \qquad (17)$$

where $\mathbf{u}$ is a T x N vector of all asset errors for all time periods.

It is straightforward to show (see Appendix) given the interpretation of $Tr(\mathbf{uu'})$ as the equal-weighted average of the error variance pooled over all time periods and assets, that the R-squared of a pooling regression of equation (14) over all periods and assets is identical to the weighted average R-squared of the separate time-series regressions of all assets ($R^2_{AVG}=\sum_{i=1}^{N} w_i R_i^2$,with $w_i=\sigma_i^2/\sum_{i=1}^{N}\sigma_i^2$, where R_i^2 is the time-series R-squared and σ_i^2 the return variance of asset i), which may be computed directly as:

$$R^2_{AVG}=Tr[\mathbf{Z}(\mathbf{Z}'\boldsymbol{\Sigma}^{-1}\mathbf{Z})^{-1}\mathbf{Z}']/Tr(\boldsymbol{\Sigma}) \qquad (18)$$

If we substitute $\mathbf{S}=\boldsymbol{\Sigma}^{-1}\mathbf{Z}$ from equation (15) or $\mathbf{S}=\boldsymbol{\Sigma}^{-1}\mathbf{Z}(\mathbf{Z}'\boldsymbol{\Sigma}^{-1}\mathbf{Z})^{-1}$ we obtain alternatively:

$$R^2_{AVG}=Tr[\boldsymbol{\Sigma}\mathbf{S}(\mathbf{S}'\boldsymbol{\Sigma}\mathbf{S})^{-1}\mathbf{S}'\boldsymbol{\Sigma}]/Tr(\boldsymbol{\Sigma}). \qquad (19)$$

Either expression may be employed to evaluate the explanatory power for time-series variation of a set of factors or the CMPs associated with a set of characteristics. [3]

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[3] The scaling of the characteristics (by a possibly different proportion for each characteristic), so that we obtain $\mathbf{ZD_Z}$, where $\mathbf{D_Z}$ is an invertible K x K diagonal scaling matrix, or scaling of the portfolio weights of the factors, so that we obtain $\mathbf{SD_S}$, where $\mathbf{D_S}$ is an invertible K x K diagonal scaling matrix, has no impact on the R-square measure in equations (18) or (19), as is easy to derive using these equations.

Empirical Testing Procedure

It may be that the CMP due to random variation explains more or less of return variance. To adjust for this we provide a formal statistical test. The right-hand side of eq. (17) is in the form of a generalized Rayleigh Quotient (see for instance Li, 2015) and must be positive and no larger than the sum of the K largest eigenvalues of $\mathbf{\Sigma}$ (as well as no smaller than the sum of the K smallest eigenvalues of $\mathbf{\Sigma}$). Thus, we propose a test to distinguish characteristic and loading formulations entirely based on the error variance: compare the trace in eq. (17) for actual data to a critical value. By this method we adjust for the fact that any tradable factor will naturally explain its own variance, and also take into account the specifics of the cross-sectional distribution of each characteristic that may otherwise affect the outcome.

Permuting Characteristics

From eq. (17) or (18) we focus on $Tr[\mathbf{Z}(\mathbf{Z}'\mathbf{\Sigma}^{-1}\mathbf{Z})^{-1}\mathbf{Z}']$ as our measure of additional variation explained by the risk factor as opposed to the characteristic. To provide a distribution under the *null hypothesis that the CMP explains no additional variation in returns*, we propose to use the original distribution of cross-sectional characteristics $\mathbf{Z}$ but permuted cross-sectionally for some of the characteristics: $\mathbf{Z_P} = [\mathbf{Z_1}\ \mathbf{PZ_2}]$, where we consider the significance of the CMPs from the set of characteristics $\mathbf{Z_2}$ (which could include anywhere between all of $\mathbf{Z}$ or as little as one column). Keeping the characteristics in $\mathbf{Z_1}$ constant, we randomly change the order of the characteristics as they are assigned to the different assets by multiplying $\mathbf{Z_2}$ by the (orthogonal) permutation matrix $\mathbf{P}$ which randomly permutes the order of the rows of $\mathbf{Z_2}$. The resulting CMPs are given as

$$\mathbf{r_t^{Z_P}} = \mathbf{S_P}'\mathbf{r_t} = (\mathbf{\Sigma}^{-1}\mathbf{Z_P})'\mathbf{r_t} = [\mathbf{\Sigma}^{-1}\mathbf{Z_1}\ \ \mathbf{\Sigma}^{-1}\mathbf{PZ_2}]'\mathbf{r_t} = \begin{pmatrix} \mathbf{Z_1}'\mathbf{\Sigma}^{-1}\mathbf{r_t} \\ \mathbf{Z_2}'\mathbf{P}'\mathbf{\Sigma}^{-1}\mathbf{r_t} \end{pmatrix} = \begin{pmatrix} \mathbf{r_t^{Z_1}} \\ \mathbf{r_t^{Z_{2P(Z)}}} \end{pmatrix}. \qquad (20)$$

Clearly, the permutation results in changing only the CMPs related to the second group of characteristics which are to be assessed. Apart from aspects of the original distribution of characteristics which remain unchanged, the CMPs based on the permuted characteristics should have no inherent explanatory power for return variations and accordingly are an appropriate benchmark for the null hypothesis that the original CMPs based on $\mathbf{Z_2}$ explain no additional variation in returns.

Accordingly: when treating risk factors derived as CMPs from characteristics we permute the appropriate part of the $\mathbf{Z}$ matrix to establish a benchmark to evaluate the null hypothesis that a particular set of factors explain no additional time-series variation. We can then straightforwardly calculate the distribution of the average time-series R-squared based on the permuted cases by computing $Tr[\mathbf{Z}(\mathbf{Z}'\mathbf{\Sigma}^{-1}\mathbf{Z})^{-1}\mathbf{Z}'] / Tr(\mathbf{\Sigma})$ for each permutation.

6. A Powerful Test for Distinguishing Risk Factor and Characteristics Explanations

The construction of the CMP makes it particularly useful for deciding if a variable converted to a factor has significant explanatory power for time-series differences in returns. The criterion is the variance-weighted average across all test assets of the time-series R-squared as given in equation (18). It is not the absolute level of this R-squared that matters since certain factor formulations automatically generate higher R-squared that may have spurious explanatory power. What matters is the level of the R-squared relative to the R-squared for similar alternative cases based on random draws. By construction, the CMP is informative about the alternative hypothesis of a true underlying factor because it maximizes the exposure to the characteristic that represents factor exposure and has the maximum signal about the underlying factor. On the other hand, by construction in the "dual" form (the dual of the formulation in equations 1), the CMP may be viewed instead as minimizing its factor variance subject to a given level of exposure to the characteristic. Lower factor variance generally reduces time-series explanatory power. Accordingly, a proper test should compare against similar type factors.

Simulation Design

Bootstrapping simulations are constructed as follows. We utilize the actual excess returns of our main test assets, the Fama-French 30 monthly industry portfolios for the period 1963.07 to 2017.12. We create an artificial factor by adding to the existing returns the product of a random (but constant over time) factor loading for each asset and a random factor realization for each period. The random factor is drawn from an i.i.d. normal distribution calibrated so that the mean annual factor return is 5 percent and the annual Sharpe Ratio is 0.35. The factor loadings are based on uniform random draws of a set of characteristics (one for each test asset) plus normally distributed random noise with zero mean, under different parameters and specification so that we evaluate the power of the test for various correlations between factor loadings and characteristics.

The idea is to capture the important practical consideration that (under the alternative hypothesis) characteristics are viewed as proxies for asset loadings on some unobserved risk factor. Thus,

$$\mathbf{r}_t^{\mathbf{SIM}} = \mathbf{r}_t + \mathbf{b}\, r_t^{F,SIM} , \tag{21}$$

$$r_t^{F,SIM} = \mu + \sigma\, \varepsilon_t^{F,SIM} , \qquad \varepsilon_t^{F,SIM} \sim N(0,1) . \tag{22}$$

$$b_i = a[c(\gamma_i Z_i) + \omega_i] , \quad \omega_t \sim N(0,1) ,\ \gamma_i \sim 2U(0,1) \text{ or } \gamma_i = 1 ,\ Z_i \sim \sqrt{12}\, U(0,1) . \tag{23}$$

Here **b** is the vector of b_i for *i* from 1 to N. The $\mathbf{r}_t^{\mathbf{SIM}}$ are the simulated returns for each period *t* based on the true returns $\mathbf{r}_t$ and adding to it a random factor realization $r_t^{F,SIM}$ drawn from a normal distribution with mean μ and standard deviation σ (such that the mean annual factor return is 5 percent and the annual Sharpe Ratio is 0.35), multiplied by factor loadings b_i for each asset drawn from the product of two independent uniformly distributed variables γ_i, Z_i on the multiples of the interval (0,1). The multiples are set so that (1) $E(\gamma_i) = 1$ and (2) $\text{var}(Z_i) = 1$ The product $c\gamma_i Z_i$ is added to a standard normally distributed variable ω_i . The parameter *c* varies across the simulations and may be viewed as a signal-to-noise parameter that controls the correlation between the characteristics Z_i and the loadings b_i. Parameter *a* is chosen so that the true (in simulation) time-series R-squared is equal to a particular fraction (typically 2%, which is roughly equal to the variance explained by the fifth or sixth eigenvector of the 30 monthly industry portfolios for the period 1963.07 to 2017.12, see Table 1).

Based on equations (21)-(23) drawn for *t = 1 to T* where *T* = 654, the number of months in our sample, we have one simulated data set for which we know the exact impact of a factor that is moderately important. We draw *K = 1000* versions of this data set. For each version of the data set, we construct a factor from knowledge of the characteristics Z_i using either the Fama-French Factor Mimicking Portfolio (FMP) approach or our CMP approach. Then calculate the average time-series R-squared based on either equation (18) for the CMP approach or equation (19) for the FMP approach. Or, equivalently, running time-series regressions of the returns of each of the *N = 30* test assets in turn on the constructed CMP factor or FMP factor and averaging (using variance weights) across all 30 assets.

For each data set, we then compare the obtained R-squared (separately for the CMP and FMP cases) against a benchmark distribution based on randomly chosen characteristics. Instead of the set $\{Z_i\}$ that directly affects the factor loadings b_i and thus explains the simulated returns $\mathbf{r_t^{SIM}}$ we *permute* this set to generate $\{\mathrm{P_j} Z_i\}$ so that the characteristics maintain their distribution but are no longer associated with the realized returns. We do this for $j = 1$ *to* J where $J = 1000$, and calculate the time series R-squared in each case, to generate a distribution under the null hypothesis that the characteristics and associated factors (CMP or FMP) have no explanatory power in the time series. If the R-squared for the original, true, set of factors obtained from $\{Z_i\}$ is larger than 95% of the R-squareds for the permuted characteristics $\{\mathrm{P_j} Z_i\}$ then we reject the null hypothesis for this data set and conclude that the CMP or FMP has real time-series explanatory power and should be interpreted as a risk factor.

We do this for all K different simulated data sets and find the fraction of times that the test rejects the null hypothesis. Since in that fraction of the cases we reject the null hypothesis when it is indeed false, we call this fraction the *power* of the test.

For each draw of a factor and associated characteristics and factor loadings and using either the FMP approach or the CMP approach to generate observable factors from observed characteristics, we construct a bootstrapped distribution by randomly permuting across the $N=30$ assets $J=1000$ times the characteristics (the assignment of the characteristics to each asset are scrambled) and then calculate the R-squared in each case. For each of the $K=1000$ factor draws we decide if the R-squared is larger than 95% (or 90% or 99%) of the R-squareds of the 1000 permutations. If so, we reject the null hypothesis of no-explanatory-power for the time-series. The fraction of these rejections provides the empirical *power* of the test because the rejections are correct since the generated factors have inherent explanatory power. We can use the same setup but create returns for which factor loadings do not depend on characteristics. In that case, the fraction of rejections provides the empirical *size* of the test.

Simulation Results

The size of the tests for both the CMP and the FMP approaches given the 95% level of significance is in all cases very close to 5%. This is not surprising given that we simply construct characteristics that are random and statistically unrelated to realized returns. Then when we permute the characteristics across the different assets these characteristics are also statistically unrelated to realized returns so that the R-squareds associated with the un-permuted characteristics stochastically exceed the 95% critical values around 5% of the time, irrespective of whether we use the FMP approach or the CMP approach to generate factors from the characteristics. The results are not tabulated but available from the authors.

We focus here on the power of the test under the two different approaches. Table 2 provides the results for a benchmark case in which we consider the normalized returns on 30 portfolios over 654 periods such that the covariance matrix before adding impact of the simulated factor is the *identity matrix*. In this case, a true factor would be relatively easy to identify and we expect high power for both approaches. We also ignore the multiplicative randomness in the factor loadings ($\gamma_i = 1$ for all *i*), further simplifying identification of a true factor. The signal-to-noise coefficient varies from 0 to 2. As Table 2 indicates, the power is around 5% for both approaches when the signal-to-noise coefficient *c* equals zero. This effectively provides the size since, although a relevant risk factor exists, its factor loadings are by design unrelated to the stipulated observable characteristic. As the signal-to-noise ratio increases, the factor loadings become increasingly tightly related to the characteristics, and the power of the test for both the CMP and the FMP approaches increases from 5% to 100% as the signal-to-noise parameter increases from zero to one. Table 2 also shows that the correlation between the factor loadings and the observable characteristics increases from zero to close to 90%.

The "true" R-squared for the simulated factor of 2% based on appropriately choosing parameter *a* in equation (23), listed in the table as "actual" R-squared, is for the simulated data indeed very close to 2% for all values of *c* but could be calculated only if one knew the latent factor realizations. The observed R-squareds for the mimicking factors generated by either the CMP or the FMP approaches may be higher or lower than the "true" R-squared. Higher is possible because the mimicking factors are tradable and thus explain a linear combination of the asset returns perfectly by design. Table 2 shows that the CMP and FMP average R-squareds given the permuted ("useless") characteristics are around 3 to 4% (increasing slightly for the CMP and decreasing slightly for the FMP as *c* increases). However, the R-squareds clearly increase as *c* increases for the un-permuted ("useful") characteristics, doubling approximately for the CMP and increasing by about 50% for the FMP. The change in the R-squareds for the useless characteristics cases is related to the fact that the factor impacts and return variances change as the signal-to-noise parameter c changes and parameter a changes in response to keep the "true" R-squared constant.

Table 3 provides the relevant results as it assumes the covariance matrix before adding impact of the simulated factor is the *actual covariance matrix* for the 30 industry portfolios for monthly excess returns over the period from 1963.07 - 2017.12. The results are striking in comparison to the benchmark in Table 2. The power for the CMP approach is still high, rising from close to 5% at $c=0$ to 100% at $c=2$. For $c=1$ the power is 65% and the correlation between factor loadings and characteristics is 70%. The R-squared for useless characteristics is 1.4% and for useful characteristics 2.5%. On the other hand, the R-squareds for the FMP case for both useless and useful characteristics are substantially higher at around 7% (and, in fact, higher for the useless characteristics case). However, the power in the FMP case is extremely low here, actually decreasing from 5% for $c=0$ to 3% for $c=2$. Thus, in practice we consider the FMP approach inadequate for deciding if a variable may be viewed as a factor or as a characteristic.

There are several observations about the number in Table 3 that are interesting. First, the FMP approach produces factors with naturally higher time-series R-squareds compared to the CMP approach. The reason follows from the optimization in section 4, namely that the CMP factors are chosen to minimize factor variance subject to a given level of exposure to the characteristic. By design, such a factor explains less of portfolio return variance. However, it also by design has a higher signal-to-noise content (high exposure to the characteristic relative to its return variance) which causes it to outperform random alternatives. An additional observation is that the performance of the FMP approach decreased dramatically in moving from uncorrelated returns with similar variances to a realistic covariance case. The reason is that the return covariances make it more difficult to pick up the signal about the latent factor from the characteristics. Note that the FMP R-squared increases with c for the useful characteristics case, but very slowly. On the other hand, the CMP approach utilizes the inverse covariance matrix in designing the factor from characteristics which helps to offset the return covariances and isolate the true signal about returns from the characteristics.

Table 4 explores the impact of the importance of the factor in explaining time-series fluctuations. In the previous tables we set the explained R-squared (if the true factor were known) equal to

2%. Here we examine the impact of *different values for the explained R-squared*. We evaluate these setting *c=1* and maintaining the realistic covariance matrix. We consider the R-squared for the introduced factor for the cases of 0.5% , 1%, 2%, 4%, 8%, 16%, and 32%. Table 4 shows that for all cases the correlation between characteristics and factor loadings is 70% (since *c* is constant). The power for CMP increases monotonically with the explained R-squared from around 10% to 81%, and the power for FMP increases from around 5% to 70%. The power for CMP is higher than that for FMP for each realized R-squared but the difference diminishes as the R-squared rises and for R-squared = 32% is relatively small at 81% vs. 70%. In addition, at this high R-squared the average FMP R-squared (for useful characteristics) is now 29% while that for CMP is only around 4%. Thus, for high (maybe unrealistically high) explanatory factor of a latent factor, the FMP approach becomes competitive with the CMP factor.

Table 5 maintains similar parameters as Table 3 but also allows random variation in the component multiplying the characteristics in addition to the additive random component added to characteristics in determining the loadings. The objective is to see if our results above are sensitive to the specific formulation of the relationship between loadings and characteristics (for given correlation, say). The power for CMP is now increased even though the correlation between characteristics and loadings is similar as in Table 3. For *c*=0.5 the power is 26%; for *c*=1 the power is 71%; for *c*=1.5 the power is 92%; and for *c*=2.0 the power is 97%. In all cases, again the power for FMP hovers around the 5% level.

Table 6 maintains the parameters as in Table 3 but now adds the market CMP as a guaranteed factor. Including the market is quantitatively important since the market factor, being often quite similar to the first eigenvector of the test assets, generally has substantial explanatory power for the time-series of the test assets, and many candidate factors are strongly correlated with it.

As Table 6 shows, including the market factor increases the power for the CMP approach which is now 61% for *c*=1 and 88% for *c*=2. While the average FMP R-squared for all *c* values is now much higher due to inclusion of the market (from 63% to 64%) and is again above the average CMP R-squared (which now varies from 60% to 61%), the power for the FMP approach is again negligible, everywhere around 5%.

In summary, in some circumstances both the CMP and FMP approaches have high power to discern a true factor and distinguish a risk factor from a characteristic effect, but the FMP approach is only competitive with the CMP approach for cases that may not be practically relevant – when returns are mostly uncorrelated or when the latent factor explains a very large fraction of time-series variation. On the other hand, the CMP approach always has substantial power as long as the correlation between the observed characteristics and the factor loadings on the unobserved factor are reasonably high. It should be emphasized that the consistently higher R-squared for the FMP approach is an indicator of spurious explanatory power and emphasizes the importance of comparing the R-squared against a background distribution.

7. Empirically Differentiating Risk Factor and Characteristics Explanations

We target our methodology to standard cases in which conflicting characteristics and risk factor explanations have been proposed. The most straightforward application is to consider the factors and characteristics associated with the Fama and French (2015) five-factor model. These concern size, value, profitability, and asset growth characteristics, in addition to market covariances. As our test assets we will be concerned with the 30 equal-weighted industry portfolios constructed by Fama and French augmented with the five factor portfolios from Fama and French (2015). The industry portfolios are interesting because they are diversified but have not first been sorted based on specific characteristics with controversial pricing impact. We add the five factor portfolios in part because this is recommended by Lewellen, Nagel, and Shanken (2010) to increase the challenge but also because it allows a direct comparison between our approach and that of Fama and French while keeping all factors tradable assets. The factors we consider are derived from characteristics in three ways: (a) using the CMP approach, (b) using our construction of Fama-French factor-mimicking portfolios (FMPs), and (c) using the actual Fama-French factors (the FF approach).

Explanatory Power for Mean Return Differences

We compare the performance of factor and characteristics formulations for the 35 industry-plus-factor portfolios. We consider the actual Fama-French (2015) market, size, value, profitability, and investment factors, and the CMPs and FMPs for each of the associated characteristics. The Fama-French factors representing the systematic risk related to a characteristic in practice have been formed quite differently from the way we construct the CMP. For instance, Fama and

French (2015) find the *value risk factor* by first collapsing the value characteristic, BM (the log of book to market equity), into +1 for the 30% highest BM firms and -1 for the 30% lowest BM firms (but adjusting for size), and call the returns of this factor at each time the *HML* factor (the value factor from the FF approach). There is no clear objective basis for this particular choice. We contend that the distinction created between factor and characteristic in this fashion is not a robust and fundamental one but rather arising from differences in the technical details of the mimicking factor construction.

In contrast, as the alternative value factor, the *value-CMP* factor (the value factor from the CMP approach) for which the loading of each asset equals the asset's covariance with the factor, we use our approach to calculate $r_{\mathrm{t}}^{\mathrm{value-CMP}} = \mathbf{r_t}'\mathbf{\Sigma}^{-1}\mathrm{BM}$ (where BM represents the vector of value-measures for each test asset). While there is little theoretical reason to expect either the HML factor or the value-CMP factor to outperform the other, it is useful to consider both specifications since Kogan and Tian (2015) argue that the empirical performance of factor and characteristics formulations in explaining mean returns is quite different. In addition, we consider the *value-FMP* in which we apply the Fama-French approach based on the 30 industry portfolios. The FMP returns are given by $r_{\mathrm{t}}^{\mathrm{value\text{-}FMP}} = \mathbf{r_t}'\mathbf{s(z)}$, where $\mathbf{s(z)}$ equals +1 for the industry portfolios with the 30% highest **z** (here BM values), -1 for the industry portfolios with the 30% lowest **z**, and 0 for all other portfolios. The other characteristics are treated similarly.

The cross-sectional results for the CMP approach are in Panel A, those for the FMP approach are in Panel B, and those for the FF approach are in Panel C of Table 7. We stress, however, that whether the CMP approach, the FMP approach, or the FF approach performs better is irrelevant for the determination of whether to adopt a characteristics or a risk-factor explanation – it will just favor *one particular proxy* for measuring the value impact (or the size, profitability or investment impact) over another). While under the FF and FMP approaches of constructing factors the characteristics and the covariance between the characteristic-based factor and returns are not the same, showing that both characteristics and the characteristic-based factor matter jointly is not evidence of a characteristics effect; it is just an indication that reasonable proxies of the same attribute are sufficiently dissimilar and each contain different noisy information regarding the same underlying variable.

Table 7 shows for the 35 portfolios for the period July 1963 – December 2017 the standard test for the explanatory power of characteristics in addition to the loadings of the factor. We employ the Fama-MacBeth (1973) cross-sectional two-pass approach with the proviso that, as in Black-Jensen-Scholes (1972) the factor loadings are estimated once for the full period. We also consider the first-pass results to examine the economic and statistical significance of the alphas.

As is consistent with previous studies (e.g., Lewellen et al. 2010), the various standard models consisting of combinations of the market, value, size, profitability, and investment factors have reasonable overall explanatory power for the mean returns of the industry portfolios plus the five factors: the cross-sectional R-squareds vary from 45% for the CAPM to between 56% and 69% for the three and five factor models. However, the prices of risk are generally not significant and differ in sign and magnitude from what is expected. The GRS test (Gibbons, Ross, and Shanken, 1989) convincingly rejects the null hypothesis that the alphas (or mean pricing errors) are zero. And the absolute values of the alphas are sizeable, larger than 2% annualized.

On the whole, therefore, for these 35 test portfolios the explanatory power for mean returns of both the three and five factor models is quite weak. However, note our key point that this performance in terms of explaining *mean returns* is completely irrelevant as far as the decision is concerned of whether to interpret and use the variables as factors or as characteristics. (To that end we should only consider the "first-pass" time-series fits, as we discuss in the following).

Explanatory Power of Shocks to the Risk Factors

The only reasonable way to distinguish factor and characteristic views is to examine if the CMPs (or the factors based on the Fama-French approach) explain a significant part of the return variation over time. If so, the variables are useful for portfolio management and risk management. Table 8 reports actual and simulated time-series R^2 averaged over all 35 test assets for the period July 1963 – December 2017, for models consisting of the market factor plus subsets of four additional factors derived from the size, value, profitability, and investment characteristics. The actual average time-series R^2 for each model is calculated based on equation (19), $R^2_{AVG} = Tr[\mathbf{\Sigma S}(\mathbf{S'\Sigma S})^{-1}\mathbf{S'\Sigma}]/Tr(\mathbf{\Sigma})$. The factors are obtained in three different ways: the CMP approach by which $\mathbf{S} = \mathbf{\Sigma}^{-1}\mathbf{Z}$ (with $\mathbf{Z}$ the N x K matrix of characteristics in which the market characteristics are captured by the market return covariances, MCV, with each of the test assets);

the FMP approach by which **S** for each factor equals +1 for the industry portfolios with the 30% highest values for the characteristic, equals -1 for the industry portfolios with the 30% lowest values for the characteristic, and 0 for all other test assets; the FF approach by which **S** for each factor equals 1 for the test asset associated with each factor (market MKT, size SMB, value HML, profitability RMW, and investment CMA) and zero for all other test assets. The CMP results are in Panel A, the FF results in Panel B, and the FMP results are in Panel C.

The actual results are compared to a background distribution to establish statistical significance. We provide simulations under the null hypothesis that the factor has no explanatory power. Naturally, any random factor will register some explanatory power, in part because it will have a 100% R-squared in explaining the time series of itself. The background distribution is obtained either by *permuting* characteristics for the marginal factors or by *drawing* characteristics from a distribution that matches the first four moments of the true distribution of characteristics. We subdivide the characteristics $\mathbf{Z} = [\mathbf{Z}_1\ \mathbf{Z}_2]$ into two sets of characteristics. The first set $\mathbf{Z}_1$ is an N x K_1 matrix consisting of K_1 factors presumed to be true factors. The second set of characteristics $\mathbf{Z}_2$ is an N x K_2 matrix representing the characteristics of K_2 marginal factors to be evaluated. Random permutation of the rows of $\mathbf{Z}_2$ maintains the same distribution of characteristics but assigns the characteristics tied to the marginal factors to the wrong assets so that these characteristics are essentially useless whether or not they have an actual link to risk factors. The background distributions for the R^2s are established by simulating $J = 1{,}000$ sets of marginal characteristics $\mathbf{Z}_{2_j}$ via random permutation of the rows of $\mathbf{Z}$. Drawing from a matched distribution involves drawing 1,000 random NxK_2 matrices from a distribution that shares the same mean, variance, skewness, and kurtosis with the original $\mathbf{Z}_2$. In both the permutation and draw cases, for the CMP and FMP approaches the background distributions are based on the imputed portfolio shares $\mathbf{S}_j$ related to the $\mathbf{Z}_j = [\mathbf{Z}_1\ \mathbf{Z}_{2j}]$, using $Tr[\mathbf{\Sigma}\mathbf{S}_j(\mathbf{S}_j'\mathbf{\Sigma}\mathbf{S}_j)^{-1}\mathbf{S}_j'\mathbf{\Sigma}]/Tr(\mathbf{\Sigma})$ to calculate the distribution of average time-series R^2 under the null hypothesis that the marginal factors are not true risk factors. For the FF approach critical values are based on random permutations of the marginal factors from the **S** matrix directly, using only the permutation method. The critical values at the 50th, 90th, 95th, and 99th-percentile cutoffs from the permuting (moment-matched drawing) methods are presented in the left (right) panels.

Results for the market factor

First we establish whether the market factor (used directly or, equivalently, inferred from the market covariance, MCV, the market-based characteristic) is indeed a risk factor. For MCV by itself the R-squared is 50.85% for the CMP approaches. The critical values equal 12.21% at the 99%, and are greatly surpassed by the actual R-squared of 50.85%. Thus, there is no doubt that the market by itself is a risk factor.

It is not immediately clear, however, whether, when added to other factors (in particular, the four characteristics-based factors), the market still has marginal explanatory power for return fluctuations. To check this we assume the most extreme case that the four characteristics-based factors are all true risk factors and we confirm if the addition of the market has a significant marginal effect. The actual R-squared of all five factors combined is 53.09% when the four characteristics-based factors are obtained via the CMP method. Permuting the covariances with the market (but not the four characteristics) generates a distribution of R-squareds with 99% critical value of 19.91% so that again the market clearly has a significant marginal effect. Results are similar if the covariances with the market are drawn randomly (yielding a 99% critical value of 19.95%).

In the special case of the market factor, the Fama-French approach does not involve use of characteristics; Fama and French (1993) directly obtained MKT as the value-weighted return of all assets considered. Thus, we cannot evaluate the significance of MKT for the FF approach. For the FMP approach, the method would require holding the assets with the top 30% MCV and shorting those with the bottom 30% MCV (based on permutation of the MCV for the background distribution). The results for this FMP approach strongly support the market excess return as a factor, but are not shown in the table as the generated background distribution involves shorting stocks and accordingly is not representative of the distribution of the market excess return.

Results for single factors added to the market

The explanatory power of the factor shocks for the 30 industry portfolios combined with the five Fama-French factor portfolios is provided in Table 8 where we look at the impact of various Fama-French style factors and CMPs on the explained variance of the portfolios (the FF30+5).

We examine first the explanatory power of each factor in isolation when added to the market factor. The results are as follows.

Table 8 presents the average time-series R-squareds for models that include the market factor (captured by market covariances with the assets, MCV) as well as one of the additional factors (based on one of the characteristics). This allows us to examine the significance of the additional factor in explaining time-series variation once a large fraction of the variation has already been absorbed by MCV. The marginal significance may be assessed by considering random alternative factors (without explanatory power) while keeping MCV constant.

For the factor based on the size characteristic SZ using the CMP factor the actual R-squared is 51.94% which exceeds the critical value at the 95% level of 51.85% as shown in Panel A for the Permutation-based background distribution. Thus, size when added to the market only may be viewed as a risk factor rather than a characteristic. For the FMP approach in Panel B, the actual R-squared is 60.03% which comfortably exceeds the critical value at the 99% level of 58.49% so that here size is also a risk factor when added to the market factor. Similarly, for the FF approach in Panel C, the size-based factor (here SMB) when added to the market factor (here MKT) generates an actual R-squared of 63.13% which clearly exceeds the 99% level of 58.56% so that size is a risk factor based on the FF approach as well.

The factor based on the value characteristic BM may also be viewed as a risk factor when added to the market: The CMP approach generates an R-squared of 51.37% while the 99% critical value is 51.28%. On the other hand, for the FMP approach the actual R-squared of 54.89% is only slightly above the median for the background distribution. A possible explanation may be the lack of power of the FMP approach documented in the previous section. For the FF approach the actual R-squared of 53.57% is also not significant and, in fact, below the median of the background distribution.

The profitability (OP)-based factor is not significant when added to the market based on CMP: the R-squared of 51.23% is below the critical 95% value of 51.25% in Panel A. Thus when added to the market only, the OP variable should be considered a characteristic rather than a risk factor. Note that OP is quite close to being significant since the 90% critical value is 51.22%. The other approaches are at odds. For the FMP approach, the actual R-squared is 58.65% which exceeds the critical value at 99% of 58.52%, while for the FF approach the R-squared for the profit factor (RMW) is 52.96% which is below the median of the background distribution.

The factor based on the investment characteristic INV is significant at the 95% level in Panel A using the CMP approach with R-squared of 51.01%. For FMP and FF, however, INV is not significant.

In general, the conclusions for the draw-based background distributions match those for the permutation-based distributions with the exception of the CMP approach for INV for which the actual R-squared of 51.009 is significant at 95% level for the permutation approach (critical value of 51.008) but not significant at the 95% level for the draw approach (critical value of 51.011%).

Overall, the market factor behaves consistent with being a risk factor, representing a systematic risk. Based on the CMP approach, SZ and BM may be considered as risk factors when added to the market, whereas the evidence for OP and INV to be risk factors is mixed.

If at least one of the characteristics is considered to be a clear risk factor then the question is not if the other characteristics are risk factors when added to the market, but if they are risk factors when added to the market plus other relevant risk factors. To deal with this issue we consider next the case where some or all of the other characteristics are risk factors in addition to the market.

Results for single factors added to the market and other factors

Apart from the marginal characteristic to be considered as a possible risk factor, we consider here all other characteristics added to the market as risk factors. We include all five characteristics, keeping four constant and permuting (or drawing for) the fifth (the marginal factor). Thus we are controlling for other factors in assessing the significance of adding the marginal factor. By design we have four cases that all have the same actual R-squared but have different background distributions.

For the CMP approach in Panel A (third grouping) we find an R-squared of 53.09% when we include the market as well as the four characteristic-based factors. For SZ this is significant at the 95% level; for BM this is significant at the 99% level. OP and INV, however, are not significant.

For the FMP approach in Panel B the R-squared is 69.99% when all four characteristics are added to the market. However, likely as a result of the lack of power of the FMP approach, while above the median in all four cases, none of the four characteristics are significant risk factors. For the FF approach only the SZ-based factor (SMB) may be considered a significant risk factor when added to the market plus the other three characteristics-based factors.

We also look more specifically at whether size when added to value and market is significant or whether value when added to size and market is significant. We find that SZ is significant when added to BM and MCV (at the 95% level) and BM is significant when added to SZ and MCV (at the 99% level) for the CMP approach. For the FMP and FF approaches SZ is significant at the 99% level but BM is not significant.

Given these results we may view BM- and SZ-based factors as risk factors (at least given the CMP approach). When we add these to the market, the question is whether further adding OP- and INV-based factors adds significantly to the R-squared. As shown in Panel A (fourth grouping) neither OP nor INV adds significantly to the R-squared when added to MCV, SZ, and BM. The same is true for the FMP and FF approaches.

Based on adding single factors at the margin, we conclude that the market, size, and value-based characteristics may be viewed as risk factors, but that the profitability and investment-based characteristics are not risk factors.

Below we check if adding factors as a group changes this conclusion. We now use the discussion around equation (20) to keep one group of factors constant while permuting the marginal factors as a group.

Results for groups of factors added to the market and other factors

In Table 8 (final grouping for all panels) we consider the joint significance of groups of factors added to a particular model. The results are as follows.

Adding size and value jointly to the market increases the R-squared (that is the average time-series R-squared of all test assets) significantly at the 99% level based on CMP (same for FMP and FF). On the other hand, adding profitability and investment jointly to the market produces an R-squared only slightly above the median for CMP (same for FF, but significant at the 95% level for FMP).

Adding size and value jointly to the combination of market, profitability, and investment increases R-squared significantly at the 99% level for CMP and FF, but less than the 90% level for FMP. But adding profitability and investment to the combination of market, size, and value does not significantly increase the R-squared (for all three approaches)

Overall choice between factors and characteristics

Clearly, for the FF30+5 assets the market should be viewed as a factor rather than as a characteristic. Relying on our CMP approach, the SZ and BM characteristic also should be viewed as risk factors. Accordingly, the three factors of the Fama and French (1993) three-factor model are rightly viewed as risk factors. On the other hand, the two factors added to generate the Fama and French (2015) five-factor model cannot be interpreted as risk factors since, while they may (or may not) explain additional variation in mean returns they do not have marginal explanatory power for time-series fluctuations in returns.

8. Conclusion

In empirical asset pricing the overwhelming focus has been on identifying which factors best explain the cross-section of mean asset returns. The prevailing method for identifying such risk factors relates them to asset characteristics. The idea is that asset characteristics serve as proxies for asset loadings on unobservable factors. The approach pioneered by Fama and French (1993) takes the characteristics of a set of assets and then constructs a portfolio (FMP) that is long on assets with high values of the characteristic and short on assets with low values of the characteristic. The returns on this FMP are interpreted as realizations representing the unobservable risk factor responsible for the characteristic affecting returns.

Fama and French's FMP approach is reasonable and universally applied but has some drawbacks. First, it is ad hoc and leaves the choice to the researcher as to how many assets to hold long with which weight and how many to short with which weight. Second, it does not take the idea seriously that the characteristics represent loadings on an unknown factor. There is a correlation but the link between the FMP's loadings and the characteristics is tenuous. Third, the FMPs tend to have a high variance component that is unrelated to the underlying factors.

To provide an alternative approach for mimicking factors avoiding these drawbacks we build on the work of Fama (1976), Ferson et al. (1999), and Balduzzi and Robotti (2008) to develop characteristics mimicking portfolios (CMPs) constructed as the up-to-scaling unique portfolios that maximize exposure to characteristics for any given variance. The loadings on CMP returns are exactly equal to the characteristics (useful for the beta formulation of asset pricing models) or, for a different scaling, the covariances between the CMP returns and asset returns equal the asset characteristics (useful for the stochastic discount factor formulation of asset pricing models). These different scalings do not affect the way the CMPs explain returns. Further, the CMPs are optimized to provide the most precise information about time series fluctuations which facilitates distinguishing true risk factors from spurious ones. Lastly, CMPs imply that there is no difference for pricing purposes of using characteristics or CMP factors as the explanatory variables. If characteristics indeed are tied to factor loadings then this is, of course, expected and using the characteristics or the loadings ought to have the same effect on asset returns.

An overwhelming number of factors and characteristics has been considered for pricing financial assets. Harvey, Liu, and Zhu (2015) distinguish 113 common (systematic) factors and 212 characteristics. We argue here in part that there is no point in distinguishing factors and characteristics when it comes to pricing. Essentially, each of the 212 characteristics may be just as well modeled as a systematic risk factor; and each of the 113 systematic factors may be converted to a characteristic. Neither would impact the explanatory power for determining expected asset returns. The difference has no pricing implications of any kind as long as the mimicking factor is constructed as a CMP. Naturally this implies that the traditional approach of Jagannathan and Wang (1996) to choose between characteristics and factors by including both factor loadings and characteristics in the same regression is futile. While mimicking factors obtained by alternative methods will be distinguishable from the underlying characteristics, the distinction is an artifact of the assumed mimicking procedure which has little theoretical backing. Any difference found between the pricing impact of the factor as different from the characteristic's pricing impact is therefore an artifact of the arbitrary mimicking process and not a robust feature of asset pricing.

Of course, understanding whether expected returns are compensation for factor risk or are instead related to particular characteristics that may matter for behavioral reasons, is still tremendously important. In the first case, expected returns come at the expense of increased risk; in the second case expected returns come without cost. Our point is simply that examining how well expected returns are explained is not informative about this issue. With characteristics models explaining expected returns indistinguishably from CMP-factor-based models, a meaningful distinction can only be based on evaluating the risk implications.

Tying factors explicitly to systematic risk would support an explanation that pricing depends on factor risk. Apart from pricing, explaining risk matters also because it is relevant for portfolio choice, risk management and hedging. For all of these purposes, it is important to evaluate how well a model explains for all assets jointly the variation of asset returns over time. The criterion of the explained fraction of the time series variation of all asset returns, allows a useful distinction between characteristic- and risk-factor-based models. This is aided by considering CMP factors as a way to separate issues and focusing only on the time-series variation for which the two approaches differ. In standard methodology the focus is on explaining alphas or on second-pass performance (if not all factors are tradable). Rarely attention is paid to first-pass fits; after all they just provide a measure of the fraction of risk that is idiosyncratic, concerning which the theory is mum. We argue here, however, that the first-pass R-squareds appropriately averaged over all assets constitutes a measure for the systematic risk explained, and that evaluating a model based on how much systematic risk it explains, both determines how useful the model is, and provides the only way to distinguish factors and characteristics.

Creating return series equal to actual returns but adding a simulated factor with random realizations for which asset loadings depend on a known characteristic and a random component, we examine how the CMP and FMP factors, both constructed from the known characteristic, perform. We find that the CMP factors always have more power than the FMP factors to reject the null hypothesis that the factor has no systematic risk. The exceptions for which FMP and CMP factors have similar power are in the unrealistic cases in which, other than due to the simulated factor, the assets are uncorrelated, or when the simulated factor explains more than 30% of the time-series fluctuations of returns. When the simulated factor explains more than 2% of time-series return fluctuations, the market factor is included, and the correlation between characteristics and loadings exceeds 70%, the power of the CMP factors is always above 70%.

We employ a bootstrapping-type approach of randomly permuting characteristics to provide a benchmark distribution for testing the null-hypothesis that converting a characteristic to a risk factor (with identical pricing implications) has no improvement in time-series explanatory power. For the four characteristics considered in Fama and French (2015) – size, value, profitability, and asset growth – we examine the empirical results for the 30 industry portfolios augmented with the five factor portfolios (four characteristics-based plus the market) (each from Fama and French). Based on these test assets as an example the size, and value factors (the original Fama-French factors) are legitimate risk factors but the profitability and asset-growth factors (the new Fama-French factors) are not.

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Appendix: Proof that $Tr[\mathbf{Z(Z'\Sigma^{-1}Z)^{-1}Z'}]/Tr(\mathbf{\Sigma})$ is the variance-weighted average time-series R-squared of all assets (R^2_{AVG}).

Since $\mathbf{\Sigma S} = \mathbf{Z}$ we have

$$R^2_{AVG} = Tr[\mathbf{Z(Z'\Sigma^{-1}Z)^{-1}Z'}]/Tr(\mathbf{\Sigma}) = Tr[\mathbf{\Sigma S(S'\Sigma S)^{-1}S'\Sigma}]/Tr(\mathbf{\Sigma}) . \qquad \text{(A1)}$$

For the time series we have that

$$\mathbf{r}_{it} = \boldsymbol{\alpha}_i + \mathbf{B}_i\mathbf{r}_t^{\mathbf{Z}} + \mathbf{u}_{it}, \text{ with } \mathbf{r}_t^{\mathbf{Z}} = \mathbf{S'r_t} \text{ and } \mathbf{B}_i = \mathbf{\Sigma}_i'\mathbf{S(S'\Sigma S)^{-1}}, \qquad \text{(A2)}$$

with $\mathbf{\Sigma}_i' = [\sigma_{i1}\ \sigma_{i2}\ ...\ \sigma_{iN}]$.

The R-squared of the time-series regression in (A2) is

$$R_i^2 = \mathbf{B}_i\mathbf{\Sigma_Z}\mathbf{B}_i'/\sigma_i^2, \text{ where } \mathbf{\Sigma_Z} = \mathbf{S'\Sigma S}. \qquad \text{(A3)}$$

Substitute in $\mathbf{B}_i = \mathbf{\Sigma}_i'\mathbf{S(S'\Sigma S)^{-1}}$ yields:

$$R_i^2 = \mathbf{\Sigma}_i'\mathbf{S(S'\Sigma S)^{-1}S'\Sigma}_i/\sigma_i^2. \qquad \text{(A4)}$$

It is easy to see by inspection that

$$Tr[\mathbf{\Sigma S(S'\Sigma S)^{-1}S'\Sigma}] = \sum_{i=1}^{N}\mathbf{\Sigma}_i'\mathbf{S(S'\Sigma S)S'\Sigma}_i . \qquad \text{(A5)}$$

Hence,

$$R^2_{AVG} = Tr[\mathbf{\Sigma S(S'\Sigma S)^{-1}S'\Sigma}]/Tr(\mathbf{\Sigma}) = \sum_{i=1}^{N} w_i R_i^2 , \qquad \text{(A6)}$$

$$\text{with } w_i = \sigma_i^2/Tr(\mathbf{\Sigma}) = \sigma_i^2 / \sum_{i=1}^{N}\sigma_i^2 .$$

Thus, the following four metrics are equivalent:

(1) the explained variance pooled over all time periods and assets divided by the sum of all these variances (the pooled R-squared); (2) $Tr[\mathbf{\Sigma S(S'\Sigma S)^{-1}S'\Sigma}]/Tr(\mathbf{\Sigma})$; (3) $Tr[\mathbf{Z(Z'\Sigma^{-1}Z)^{-1}Z'}]/Tr(\mathbf{\Sigma})$ given $\mathbf{\Sigma S} = \mathbf{Z}$; and (4) the variance-weighted average time-series R-squared of all assets.

Table 1 Descriptive Statistics of Time Series Features of the Test Assets

This table reports the 10 largest eigenvalues (EVs) of the test asset excess returns between 1963.07 and 2017.12. $Tr(\mathbf{\Sigma})$ is the total variance (the sum of the variances of all test assets). The column R^2s list the fraction of the total variance explained by each associated eigenvector in time series regressions. Test portfolios are the equal-weighted monthly excess returns of the Fama-French 30 industry portfolios (FF30) and the 30 industry portfolios plus the five zero-investment factor returns, MKT, SMB, HML, RMW, and CMA (FF35) from Fama and French (2015) for the period between 1963.07 and 2017.12 (T=654).

	FF30		FF35	
	EV	R^2	EV	R^2
1	983.01	70.57	1001.60	69.62
2	113.45	8.14	113.80	7.91
3	47.82	3.43	49.16	3.42
4	40.69	2.92	42.18	2.93
5	36.61	2.63	40.12	2.79
6	23.31	1.67	23.94	1.66
7	16.29	1.17	17.26	1.20
8	14.73	1.06	15.91	1.11
9	13.80	0.99	14.47	1.01
10	10.83	0.78	11.30	0.79
$Tr(\mathbf{\Sigma})$	1392.97		1438.58	

Table 2 Benchmark Power Comparisons for Tests of Factor versus Characteristic

This table displays the power of a test to reject the null hypothesis that a latent factor constructed from a characteristic has no explanatory power for the time series of returns. The criterion is the average variance-weighted R-squared for all test assets. The power is against an alternative in which N = 30 test assets over T = 654 periods have an identity covariance matrix supplemented with a random factor realization times random loadings (equation 21). The factor realizations are drawn from a normal distribution with annualized mean return of 5% and annualized Sharpe Ratio equal to 0.35 (equation 22). Loadings on the factor are the sum of characteristics drawn from a uniform distribution with mean and variance equal to one and a standard normal errors (equation 23). The observable factors are derived in two ways from the observable random characteristics. First, using the CMP approach that derives the factor as $r_{CMP}^{F,SIM} = \mathbf{r}^{\mathbf{SIM}\prime}\mathbf{\Sigma}^{-1}\mathbf{z}$, where **z** is the vector of random characteristics z_i, $\mathbf{\Sigma}$ is the covariance matrix of the simulated returns and $\mathbf{r}^{\mathbf{SIM}}$ is the NxT matrix of simulated return realizations; and second from the FMP approach that derives the factors as $r_{FMP}^{F,SIM} = \mathbf{r}^{\mathbf{SIM}\prime}\mathbf{s(z)}$, where $s_i = 1$ if z_i is in the top 30% among the assets *i* and $s_i = -1$ if z_i is in the bottom 30%, and $s_i = 0$ otherwise. Parameters are chosen so that the "true" time-series average R-squared of explaining the returns by the latent factor equals 2%. Loadings are drawn based on equation (23) but with $\gamma_i = 1$ for all *i*. The simulation consists of K = 1000 data sets. For each data set J = 1000 random characteristics unrelated to the factor loadings are obtained by randomly permuting the characteristics that do relate to the factor loadings. These establish a benchmark distribution under the null of no explanatory power. A realized R-squared for a characteristic related to factor loadings is considered to reject the null hypothesis if it is larger than 95% of the R-squareds for random permutations of the characteristics. The power (Power CMP and Power FMP) is calculated as the total number of rejections divided by K. Parameter *c* from equation (23) controls the correlation between loadings **b** and characteristics **z,** which is measured directly by Corr (*b,z*). CMP R2 and FMP R2 indicate the average time-series R-squareds for the factors created from the characteristics using the CMP and FMP approaches, respectively. CMPperm R2 and FMPperm R2 indicate the average time-series R-squareds for the factors created from permuted (useless) characteristics using the CMP and FMP approaches, respectively. True R2 is the average time-series R-squared targeted for the latent factor. Actual R2 is the realized average time-series R-squared for the latent factor (if this factor were observable). All outcomes are stated in percentage terms.

c	0.00	0.25	0.50	0.75	1.00	1.25	1.50	1.75	2.00
Power CMP	4.75	32.08	75.54	95.74	99.60	100.0	100.0	100.0	100.0
Power FMP	5.35	19.21	57.52	86.93	96.93	99.21	99.60	99.90	99.90
Corr (*b,z*)	-1.21	22.80	43.36	58.92	69.92	77.53	82.81	86.54	89.23
CMP R2	3.31	3.56	4.02	4.39	4.63	4.79	4.88	4.95	5.00
CMPperm R2	3.31	3.45	3.68	3.85	3.95	4.01	4.05	4.07	4.09
FMP R2	3.38	3.47	3.62	3.71	3.76	3.79	3.81	3.83	3.83
FMPperm R2	3.37	3.36	3.34	3.32	3.31	3.31	3.30	3.30	3.30
Actual R2	1.98	1.97	1.97	1.97	1.98	1.98	1.98	1.98	1.98

Table 3 Power Comparisons for Tests of Factor versus Characteristic

This table displays the power of a test to reject the null hypothesis that a latent factor constructed from a characteristic has no explanatory power for the time series of returns. The criterion is the average variance-weighted R-squared for all test assets. The power is against an alternative in which N = 30 test assets over T = 654 periods have the same covariance matrix as the 30 industry portfolios for the period 1963.07 – 2017.12 supplemented with a random factor realization times random loadings (equation 21). The factor realizations are drawn from a normal distribution with annualized mean return of 5% and annualized Sharpe Ratio equal to 0.35 (equation 22). Loadings on the factor are the sum of characteristics drawn from a uniform distribution with mean and variance equal to one and a standard normal errors (equation 23). The observable factors are derived in two ways from the observable random characteristics. First, using the CMP approach that derives the factor as $r_{CMP}^{F,SIM} = \mathbf{r}^{\mathbf{SIM}}\mathbf{'}\mathbf{\Sigma}^{\mathbf{-1}}\mathbf{z}$, where $\mathbf{z}$ is the vector of random characteristics z_i, $\mathbf{\Sigma}$ is the covariance matrix of the simulated returns and $\mathbf{r}^{\mathbf{SIM}}$ is the N x T matrix of simulated return realizations; and second from the FMP approach that derives the factors as $r_{FMP}^{F,SIM} = \mathbf{r}^{\mathbf{SIM}}\mathbf{'s(z)}$, where $s_i = 1$ if z_i is in the top 30% among the assets *i* and $s_i = -1$ if z_i is in the bottom 30%, and $s_i = 0$ otherwise. Parameters are chosen so that the "true" time-series average R-squared of explaining the returns by the latent factor equals 2%. Loadings are drawn based on equation (23) but with $\gamma_i = 1$ for all *i*. The simulation consists of K = 1000 data sets. For each data set J = 1000 random characteristics unrelated to the factor loadings are obtained by randomly permuting the characteristics that do relate to the factor loadings. These establish a benchmark distribution under the null of no explanatory power. A realized R-squared for a characteristic related to factor loadings is considered to reject the null hypothesis if it is larger than 95% of the R-squareds for random permutations of the characteristics. The power (Power CMP and Power FMP) is calculated as the total number of rejections divided by K. Parameter *c* from equation (23) controls the correlation between loadings **b** and characteristics **z,** which is measured directly by Corr (*b,z*). CMP R2 and FMP R2 indicate the average time-series R-squareds for the factors created from the characteristics using the CMP and FMP approaches, respectively. CMPperm R2 and FMPperm R2 indicate the average time-series R-squareds for the factors created from permuted (useless) characteristics using the CMP and FMP approaches, respectively. True R2 is the average time-series R-squared targeted for the latent factor. Actual R2 is the realized average time-series R-squared for the latent factor (if this factor were observable). All outcomes are stated in percentage terms.

c	0.00	0.25	0.50	0.75	1.00	1.25	1.50	1.75	2.00
Power CMP	5.84	7.43	13.56	23.66	34.36	42.87	49.50	55.05	58.32
Power FMP	4.26	3.96	3.76	4.06	3.76	3.76	3.66	3.37	3.76
Corr (*b,z*)	0.00	24.14	44.47	59.74	70.51	77.94	83.10	86.75	89.39
CMP R2	2.68	2.81	3.08	3.38	3.63	3.82	3.97	4.07	4.15
CMPperm R2	2.68	2.68	2.67	2.67	2.66	2.66	2.66	2.65	2.66
FMP R2	6.45	6.41	6.33	6.28	6.24	6.22	6.21	6.20	6.20
FMPperm R2	6.46	6.46	6.47	6.47	6.48	6.48	6.48	6.49	6.49
Actual R2	2.00	2.00	1.99	1.99	1.99	1.99	1.99	1.99	1.99

Table 4 Power for Tests of Factor versus Characteristic with Varying True R-squared

This table displays the power of a test to reject the null hypothesis that a latent factor constructed from a characteristic has no explanatory power for the time series of returns. The criterion is the average variance-weighted R-squared for all test assets. The power is against an alternative in which N = 30 test assets over T = 654 periods have the same covariance matrix as the 30 industry portfolios for the period 1963.07 – 2017.12 supplemented with a random factor realization times random loadings (equation 21). The factor realizations are drawn from a normal distribution with annualized mean return of 5% and annualized Sharpe Ratio equal to 0.35 (equation 22). Loadings on the factor are the sum of characteristics drawn from a uniform distribution with mean and variance equal to one and a standard normal errors (equation 23). The observable factors are derived in two ways from the observable random characteristics. First, using the CMP approach that derives the factor as $r_{CMP}^{F,SIM} = \mathbf{r^{SIM}}'\mathbf{\Sigma}^{-1}\mathbf{z}$, where $\mathbf{z}$ is the vector of random characteristics z_i, $\mathbf{\Sigma}$ is the covariance matrix of the simulated returns and $\mathbf{r^{SIM}}$ is the NxT matrix of simulated return realizations; and second from the FMP approach that derives the factors as $r_{FMP}^{F,SIM} = \mathbf{r^{SIM}}'\mathbf{s(z)}$, where $s_i = 1$ if z_i is in the top 30% among the assets *i* and $s_i = -1$ if z_i is in the bottom 30%, and $s_i = 0$ otherwise. Loadings are drawn based on equation (23) but with $\gamma_i = 1$ for all *i*. The simulation consists of K = 1000 data sets. For each data set J = 1000 random characteristics unrelated to the factor loadings are obtained by randomly permuting the characteristics that do relate to the factor loadings. These establish a benchmark distribution under the null of no explanatory power. A realized R-squared for a characteristic related to factor loadings is considered to reject the null hypothesis if it is larger than 95% of the R-squareds for random permutations of the characteristics. The power (Power CMP and Power FMP) is calculated as the total number of rejections divided by K. Parameter *c* from equation (23) controls the correlation between loadings **b** and characteristics **z,** which is measured directly by Corr (*b,z*) and is set equal to 1 in this table. CMP R2 and FMP R2 indicate the average time-series R-squareds for the factors created from the characteristics using the CMP and FMP approaches, respectively. CMPperm R2 and FMPperm R2 indicate the average time-series R-squareds for the factors created from permuted (useless) characteristics using the CMP and FMP approaches, respectively. Actual R2 is the realized average time-series R-squared for the latent factor (if this factor were observable). All numbers are stated in percentage terms.

True R2	0.50	1.00	2.00	4.00	8.00	16.00	32.00
Power CMP	9.70	16.34	32.08	52.87	68.51	77.13	80.89
Power FMP	4.85	4.75	4.65	5.05	8.42	29.41	69.80
Corr (*b,z*)	70.15	70.15	70.15	70.15	70.15	70.15	70.15
CMP R2	2.99	3.25	3.64	4.08	4.43	4.44	3.82
CMPperm R2	2.66	2.67	2.66	2.63	2.54	2.33	1.90
FMP R2	6.30	6.20	6.18	6.61	8.59	14.51	29.06
FMPperm R2	6.59	6.55	6.49	6.42	6.48	7.34	11.45
c	1.00	1.00	1.00	1.00	1.00	1.00	1.00
Actual R2	0.50	1.00	2.01	4.01	8.01	15.97	31.81

Table 5 Power for a Multiplicative Loading-Characteristic Distortion

This table displays the power of a test to reject the null hypothesis that a latent factor constructed from a characteristic has no explanatory power for the time series of returns. The criterion is the average variance-weighted R-squared for all test assets. The power is against an alternative in which N = 30 test assets over T = 654 periods have the same covariance matrix as the 30 industry portfolios for the period 1963.07 – 2017.12 supplemented with a random factor realization times random loadings (equation 21). The factor realizations are drawn from a normal distribution with annualized mean return of 5% and annualized Sharpe Ratio equal to 0.35 (equation 22). Loadings on the factor are the sum of characteristics drawn from a uniform distribution with mean and variance equal to one times a uniformly distributed random variable with (0,1) support and a standard normal errors (equation 23). The observable factors are derived in two ways from the observable random characteristics. First, using the CMP approach that derives the factor as $r_{CMP}^{F,SIM} = \mathbf{r}^{\mathbf{SIM}}\mathbf{'}\mathbf{\Sigma}^{-1}\mathbf{z}$, where **z** is the vector of random characteristics z_i, $\mathbf{\Sigma}$ is the covariance matrix of the simulated returns and $\mathbf{r}^{\mathbf{SIM}}$ is the NxT matrix of simulated return realizations; and second from the FMP approach that derives the factors as $r_{FMP}^{F,SIM} = \mathbf{r}^{\mathbf{SIM}}\mathbf{'s(z)}$, where $s_i = 1$ if z_i is in the top 30% among the assets *i* and $s_i = -1$ if z_i is in the bottom 30%, and $s_i = 0$ otherwise. Parameters are chosen so that the "true" time-series average R-squared of explaining the returns by the latent factor equals 2%. Loadings are drawn based on equation (23) but with $\gamma_i = 1$ for all *i*. The simulation consists of K = 1000 data sets. For each data set J = 1000 random characteristics unrelated to the factor loadings are obtained by randomly permuting the characteristics that do relate to the factor loadings. These establish a benchmark distribution under the null of no explanatory power. A realized R-squared for a characteristic related to factor loadings is considered to reject the null hypothesis if it is larger than 95% of the R-squareds for random permutations of the characteristics. The power (Power CMP and Power FMP) is calculated as the total number of rejections divided by K. Parameter *c* from equation (23) controls the correlation between loadings **b** and characteristics **z,** which is measured directly by Corr (*b,z*). CMP R2 and FMP R2 indicate the average time-series R-squareds for the factors created from the characteristics using the CMP and FMP approaches, respectively. CMPperm R2 and FMPperm R2 indicate the average time-series R-squareds for the factors created from permuted (useless) characteristics using the CMP and FMP approaches, respectively. True R2 is the average time-series R-squared targeted for the latent factor. Actual R2 is the realized average time-series R-squared for the latent factor (if this factor were observable). All outcomes are stated in percentage terms.

c	0.00	0.25	0.50	0.75	1.00	1.25	1.50	1.75	2.00
Power CMP	6.04	9.11	25.84	49.11	71.09	85.35	91.88	95.54	97.43
Power FMP	5.74	5.35	4.95	4.65	3.96	4.36	4.36	3.47	3.66
Corr (*b,z*)	-0.34	23.83	44.33	59.73	70.56	78.02	83.19	86.84	89.47
CMP R2	60.10	60.24	60.53	60.85	61.11	61.32	61.47	61.58	61.66
CMPperm R2	60.10	60.20	60.40	60.57	60.69	60.78	60.83	60.87	60.90
FMP R2	62.62	62.95	63.40	63.64	63.77	63.84	63.88	63.91	63.93
FMPperm R2	62.67	62.93	63.32	63.57	63.72	63.80	63.85	63.88	63.90
True R2	1.98	1.99	2.00	2.01	2.01	2.01	2.02	2.02	2.02

Table 6 Power for Tests of Factor versus Characteristic with Market Factor

This table displays the power of a test to reject the null hypothesis that a latent factor constructed from a characteristic has no explanatory power for the time series of returns. The criterion is the average variance-weighted R-squared for all test assets. The power is against an alternative in which N = 30 test assets over T = 654 periods have the same covariance matrix as the 30 industry portfolios for the period 1963.07 – 2017.12 supplemented with a random factor realization times random loadings (equation 21). The factor realizations are drawn from a normal distribution with annualized mean return of 5% and annualized Sharpe Ratio equal to 0.35 (equation 22). Loadings on the factor are the sum of characteristics drawn from a uniform distribution with mean and variance equal to one and a standard normal errors (equation 23). The observable factors are derived in two ways from the observable random characteristics. First, using the CMP approach that derives the factor as $r_{CMP}^{F,SIM} = \mathbf{r}^{\mathbf{SIM}}{}'\mathbf{\Sigma}^{-1}\mathbf{z}$, where $\mathbf{z}$ is the vector of random characteristics z_i, $\mathbf{\Sigma}$ is the covariance matrix of the simulated returns and $\mathbf{r}^{\mathbf{SIM}}$ is the N x T matrix of simulated return realizations; and second from the FMP approach that derives the factors as $r_{FMP}^{F,SIM} = \mathbf{r}^{\mathbf{SIM}}{}'\mathbf{s(z)}$, where $s_i = 1$ if z_i is in the top 30% among the assets *i* and $s_i = -1$ if z_i is in the bottom 30%, and $s_i = 0$ otherwise. The market factor is the value-weighted average of the industry portfolio returns. The market characteristic used for constructing the market CMP is the covariance between portfolio return and market return. Parameters are chosen so that the "true" time-series average R-squared of explaining the returns by the latent factor equals 2%. Loadings are drawn based on equation (23) but with $\gamma_i = 1$ for all *i*. The simulation consists of K = 1000 data sets. For each data set J = 1000 random characteristics unrelated to the factor loadings are obtained by randomly permuting the characteristics that do relate to the factor loadings. These establish a benchmark distribution under the null of no explanatory power. A realized R-squared for a characteristic related to factor loadings is considered to reject the null hypothesis if it is larger than 95% of the R-squareds for random permutations of the characteristics. The power (Power CMP and Power FMP) is calculated as the total number of rejections divided by K. Parameter *c* from equation (23) controls the correlation between loadings **b** and characteristics **z,** which is measured directly by Corr (*b,z*). CMP R2 and FMP R2 indicate the average time-series R-squareds for the factors created from the characteristics using the CMP and FMP approaches, respectively. CMPperm R2 and FMPperm R2 indicate the average time-series R-squareds for the factors created from permuted (useless) characteristics using the CMP and FMP approaches, respectively. True R2 is the average time-series R-squared targeted for the latent factor. Actual R2 is the realized average time-series R-squared for the latent factor (if this factor were observable). All outcomes are stated in percentage terms.

c	0.00	0.25	0.50	0.75	1.00	1.25	1.50	1.75	2.00
Power CMP	6.83	9.21	20.40	41.19	60.79	74.16	81.39	85.94	87.82
Power FMP	5.64	5.64	5.15	4.85	4.65	4.16	4.46	4.26	4.36
Corr (*b,z*)	-0.01	23.40	42.80	56.38	65.30	71.13	75.01	77.68	79.56
CMP R2	60.08	60.21	60.48	60.75	60.96	61.11	61.22	61.29	61.35
CMPperm R2	60.08	60.18	60.37	60.52	60.62	60.69	60.73	60.76	60.78
FMP R2	62.66	63.00	63.42	63.65	63.76	63.83	63.86	63.89	63.90
FMPperm R2	62.65	62.91	63.29	63.52	63.65	63.72	63.76	63.79	63.81
Actual R2	2.01	2.01	2.00	2.00	2.00	2.00	2.00	2.00	2.00

Table 7 Model Performance in Explaining Mean Returns

This table reports the Fama-MacBeth cross sectional regressions in the left panel and GRS time-series regression results are reported in the right panel. Test portfolios (FF35) are the equal-weighted monthly excess returns of the Fama-French 30 industry portfolios plus the five zero-investment factor returns (MKT, SMB, HML, RMW, and CMA) from Fama and French (2015) for the period between 1963.07 and 2017.12 (T=654). MCV is the covariance between portfolio returns and MKT, estimated during the same sample period. The four other characteristics are the time-series averages of the log(market capitalization) (SZ) and the book-to-market ratio (BM) were provided by Kenneth French's website along with the corresponding returns; operating profitability (OP) and investment (INV) were constructed according to the FF30 industry definitions. The characteristic-mimicking portfolios (CMPs) are formed as $\mathbf{\Sigma}^{-1}\mathbf{Z}$ where $\mathbf{Z}$ represents either one of the four characteristics (SZ, BM, OP, INV) and $\mathbf{\Sigma}$ is the covariance matrix of the asset excess returns. CMP returns are constructed by multiplying the FF35 excess returns to each of the CMPs.The factor-mimicking portfolios (FMPs) are formed according to Fama-French style market neutral portfolios. The size-based portfolio is formed by assigning 1s (-1s) to the smallest (biggest) 30% of the 30 industry portfolios based on SZ. The value-based portfolio is formed by assigning 1s (-1s) to the 30% of the 30 industry portfolios with the highest (lowest) BM. The profitability-based portfolio is formed by assigning 1s (-1s) to the 30% of the industry portfolios with the highest (lowest) OP. The investment-based portfolio is formed by assigning 1s (-1s) to the industry portfolios with the lowest (highest) INV. For each the weight on the five Fama-French factors is set to 0s. The FMP returns are constructed by weighting the FF35 returns with each of the FMPs. The left part of panels A, B, and C report the average of the cross-sectional regression estimates (the prices of covariance risk, are reported in the first line and the Black-Jensen-Scholes t-statistics are reported on the next line). R^2 is obtained in a single cross-sectional regression by regressing averages of asset excess returns on characteristics directly (Panel A), on the covariances between portfolio excess returns and each of the FMP returns (Panel B), and on the Fama-French risk factors (Panel C). The right sides of panels A, B, and C reports the GRS F-statistics, the p-value with numerator degrees of freedom of T-N-K and denominator degrees of freedom of N-K, and the mean absolute alphas of each time-series regressions. K represents the number of factors involved in the regressions and N is the number of test portfolios here equal to 35.

Panel A (CMP Factors)

Cons	MCV	SZ	BM	OP	INV	R2	GRS-F	p-value	Abs alpha
0.370	0.020					0.452	4.961	0.000	0.225
5.959	1.871								
0.302	0.003	0.074	-0.103			0.642	4.284	0.000	0.263
6.081	0.221	2.829	-0.963						
0.284	0.004	0.086	-0.131	-0.084	-0.322	0.659	2.519	0.000	0.256
3.993	0.252	3.092	-1.130	-0.587	-1.138				

Panel B (FMP Factors)

Cons	MCV	SZ	BM	OP	INV	R2	GRS-F	p-value	Abs alpha
0.370	0.020					0.452	4.961	0.000	0.225
5.959	1.871								
0.381	0.008	-0.005	0.000			0.640	4.970	0.000	0.310
6.642	0.598	-1.915	0.171						
0.397	0.017	-0.003	-0.002	0.003	-0.000	0.694	5.240	0.000	0.365
8.592	0.689	-0.825	-0.794	1.177	-0.054				

Panel C (Fama-French Factors)

Cons	MKT	SMB	HML	RMW	CMA	R2	GRS-F	p-value	Abs alpha
0.370	0.020					0.452	4.961	0.000	0.225
5.959	1.871								
0.434	0.025	-0.019	-0.020			0.497	4.588	0.000	0.182
8.653	2.295	-1.131	-1.075						
0.355	0.041	-0.013	-0.081	0.006	0.140	0.561	3.567	0.000	0.207
5.920	2.405	-0.707	-1.901	0.191	1.617				

Table 8 Significance of Additional Factors in Explaining Time-Series Fluctuations

This table reports the actual and simulated time-series R^2 averaged over all test assets for models consisting of the market factor plus variations of 4 additional factors derived from the size (log of market capitalization SZ), value (book-to-market ratio BM), profitability (operating profit OP), and investment (net investment INV) characteristics. The test assets are the equal-weighted monthly excess returns of the Fama-French 30 industry portfolios together with the 5 Fama and French (2015) factor portfolios for the period from 1963.07 to 2017.12. The actual average time-series R^2 for each model is calculated based on $R^2_{AVG} = Tr[\boldsymbol{\Sigma}\mathbf{S}(\mathbf{S}'\boldsymbol{\Sigma}\mathbf{S})^{-1}\mathbf{S}'\boldsymbol{\Sigma}]/Tr(\boldsymbol{\Sigma})$, where **S** represents a N x K (number of factors by number of test assets) matrix with the portfolio weights of each of the traded factors on the 35 test assets. $\boldsymbol{\Sigma}$ is the covariance matrix of the test asset returns. The factors are obtained in three different ways: the CMP approach by which $\mathbf{S} = \boldsymbol{\Sigma}^{-1}\mathbf{Z}$ (with **Z** the N x K matrix of characteristics where the market characteristics are captured by the market return covariances, MCV, with each of the test assets); the FF approach by which **S** for each factor equals 1 for the test asset associated with each factor (market MKT, size SMB, value HML, profitability RMW, and investment CMA) and zero for all other test assets; the FMP approach by which **S** for each factor equals +1 for the industry portfolios with the 30% highest values for the characteristic, equals -1 for the industry portfolios with the 30% lowest values for the characteristic, and 0 for all other test assets. The CMP results are in Panel A, the FF results in Panel B, and the FMP results are in Panel C. The actual results are compared to a background distribution to establish statistical significance. The background distribution is obtained in two different ways. First, based on permutations of the characteristics determining the marginal factors. To this end we subdivide the characteristics $\mathbf{Z} = [\mathbf{Z}_1\ \mathbf{Z}_2]$ into two sets of characteristics. The first set $\mathbf{Z}_1$ is an NxK_1 matrix consisting of K_1 factors presumed to be true factors. The second set of characteristics $\mathbf{Z}_2$ (listed in bold face in the table) is an NxK_2 matrix representing the characteristics of K_2 marginal factors to be evaluated. Random permutation of the rows of $\mathbf{Z}_2$ maintains the same distribution of characteristics but assigns the characteristics tied to the marginal factors to the wrong assets so that these characteristics are essentially useless whether or not they have an actual link to risk factors. The background distributions for the R^2s are established by simulating $J = 1{,}000$ sets of marginal characteristics $\mathbf{Z}_{2j}$ via random permutation of the rows of **Z**. Second, based on random draws of the characteristics determining the marginal factors. This method involves drawing 1,000 random NxK_2 matrices from a distribution that shares the same mean, variance, skewness, and kurtosis with the original $\mathbf{Z}_2$. In both cases, for the CMP and FMP approaches the background distributions are based on the imputed portfolio shares $\mathbf{S}_j$ related to the $\mathbf{Z}_j = [\mathbf{Z}_1\ \mathbf{Z}_{2j}]$, using $Tr[\boldsymbol{\Sigma}\mathbf{S}_j(\mathbf{S}_j'\boldsymbol{\Sigma}\mathbf{S}_j)^{-1}\mathbf{S}_j'\boldsymbol{\Sigma}]/Tr(\boldsymbol{\Sigma})$ to calculate the distribution of average time-series R^2 under the null hypothesis that the marginal factors are not true risk factors. For the FF approach critical values are based on random permutations of the marginal factors from the **S** matrix directly, using only the permutation method. The critical values at the 50th, 90th, 95th, and 99th-percentile cutoffs from the permuting (moment-matched drawing) methods are presented in the left (right) panels.

Panel A (CMP Factors)

		Permute				Draw			
	R2	50%	90%	95%	99%	50%	90%	95%	99%
MCV	50.846	7.216	9.407	10.286	12.212	7.438	10.328	11.322	13.340
SZ BM OP INV, **MCV**	53.092	15.829	17.498	18.205	19.914	15.905	17.684	18.345	19.948
MCV, **SZ**	51.935	51.608	51.789	51.848	51.968	51.617	51.866	51.953	51.935
MCV, **BM**	51.368	51.139	51.209	51.233	51.276	51.142	51.211	51.236	51.280
MCV, **OP**	51.231	51.137	51.224	51.252	51.311	51.141	51.223	51.251	51.317
MCV, **INV**	51.009	51.002	51.007	51.008	51.010	51.003	51.009	51.011	51.014
MCV BM OP INV, **SZ**	53.092	52.773	52.983	53.055	53.201	52.791	53.062	53.158	53.354
MCV SZ OP INV, **BM**	53.092	52.790	52.907	52.946	53.025	52.795	52.919	52.961	53.043
MCV SZ BM INV, **OP**	53.092	53.101	53.212	53.251	53.336	53.112	53.221	53.260	53.350
MCV SZ BM OP, **INV**	53.092	53.081	53.092	53.095	53.101	53.082	53.094	53.098	53.107
MCV BM, **SZ**	52.625	52.233	52.433	52.501	52.654	52.245	52.512	52.600	52.805
MCV SZ, **BM**	52.625	52.338	52.449	52.486	52.557	52.344	52.462	52.501	52.576
MCV SZ BM, **OP**	52.915	52.919	53.020	53.057	53.131	52.927	53.031	53.068	53.150
MCV SZ BM, **INV**	52.794	52.783	52.793	52.796	52.800	52.784	52.796	52.800	52.808
MCV, **SZ BM**	52.625	51.993	52.205	52.277	52.418	51.998	52.279	52.375	52.561
MCV, **OP INV**	51.407	51.315	51.410	51.441	51.502	51.325	51.414	51.447	51.520
MCV OP INV, **SZ BM**	53.092	52.534	52.749	52.821	52.969	52.532	52.821	52.921	53.126
MCV SZ BM, **OP INV**	53.092	53.094	53.206	53.246	53.333	53.106	53.221	53.261	53.357

Panel B (FMP Factors)

		Permute				Draw			
	R2	50%	90%	95%	99%	50%	90%	95%	99%
MCV, **SZ**	60.034	54.814	56.849	57.405	58.485	54.839	56.837	57.358	60.034
MCV, **BM**	54.887	54.800	56.825	57.349	58.519	54.813	56.848	57.395	58.554
MCV, **OP**	58.644	54.795	56.821	57.418	58.622	54.789	56.803	57.331	58.480
MCV, **INV**	55.770	54.807	56.855	57.391	58.524	54.796	56.831	57.373	58.487
MCV BM OP INV, **SZ**	69.991	69.124	70.215	70.506	71.150	69.137	70.225	70.522	71.119
MCV SZ OP INV, **BM**	69.991	69.988	71.003	71.250	71.629	69.990	71.004	71.255	71.612
MCV SZ BM INV, **OP**	69.991	69.116	70.286	70.609	71.169	69.114	70.293	70.604	71.142
MCV SZ BM OP, **INV**	69.991	69.523	70.470	70.722	71.037	69.531	70.474	70.709	71.034
MCV BM, **SZ**	64.170	58.171	60.250	61.031	62.437	58.154	60.271	61.020	62.543
MCV SZ, **BM**	64.170	63.098	65.150	65.634	66.372	63.061	65.120	65.605	66.366
MCV SZ BM, **OP**	67.489	66.606	67.940	68.296	68.858	66.617	67.976	68.333	68.853
MCV SZ BM, **INV**	66.920	66.612	67.933	68.279	68.840	66.613	67.933	68.277	68.839
MCV, **SZ BM**	64.170	58.448	60.919	61.804	63.468	58.457	60.967	61.802	63.411
MCV, **OP INV**	63.205	58.442	60.915	61.783	63.371	58.471	60.906	61.725	63.480
MCV OP INV, **SZ BM**	69.991	68.378	70.017	70.478	71.393	68.344	70.000	70.445	71.342
MCV SZ BM, **OP INV**	69.991	69.015	70.353	70.688	71.290	68.997	70.367	70.710	71.324

Panel C (FF Factors)

	R2	50%	90%	95%	99%
MKT, **SMB**	63.130	54.811	56.844	57.388	58.555
MKT, **HML**	53.567	54.823	56.833	57.392	58.527
MKT, **RMW**	52.956	54.828	56.844	57.394	58.559
MKT, **CMA**	52.292	54.801	56.850	57.427	58.579
MKT HML RMW CMA, **SMB**	68.153	59.841	61.876	62.367	63.333
MKT SMB RMW CMA, **HML**	68.153	69.117	71.262	71.612	72.006
MKT SMB HML CMA, **RMW**	68.153	70.129	72.262	72.567	72.928
MKT SMB HML RMW, **CMA**	68.153	70.894	73.070	73.380	73.708
MKT HML, **SMB**	66.759	57.545	59.669	60.286	61.511
MKT SMB, **HML**	66.759	66.345	68.432	68.818	69.349
MKT SMB HML, **RMW**	67.834	69.813	71.944	72.270	72.612
MKT SMB HML, **CMA**	67.088	69.831	71.948	72.270	72.615
MKT, **SMB HML**	66.759	58.469	60.934	61.814	63.419
MKT, **RMW CMA**	54.450	58.447	60.945	61.756	63.381
MKT RMW CMA, **SMB HML**	68.153	61.740	64.172	64.981	66.518
MKT SMB HML, **RMW CMA**	68.153	72.696	74.414	74.730	75.309