Chapter VIII. Dynamic Asset Pricing–General Aspects

The objective in this chapter is to convey general insights concerning the characteristics of asset prices and returns in a dynamic environment. While much of the discussion in the finance literature has been couched in a continuous time diffusion framework, the current discussion will be discrete time. The advantages of a discrete-time approach are that it requires less technical baggage and that the formulations are more intuitive. As a result it is easier to apply the material creatively. As in most of the continuous-time-based literature, a dynamic programming approach will be employed that has the advantage of, in a sense, combining all future time periods into one. Topics discussed in this chapter are: basic properties of representative investor dynamic asset pricing models, conditions under which the CAPM applies in a multi-period economy, the Merton model, the consumption CAPM, and a discussion of asset pricing puzzles and other stylized facts.

1. BASIC PROPERTIES OF DYNAMIC ASSET PRICING MODELS

(a) A Representative Investor Model

To simplify matters initially, assume that a representative investor exists. Now consider the following decision problem. The representative investor maximizes expected utility for a time-separable infinite horizon utility function. Choice variables are the portfolio and the consumption level in each period and the constraint is lifetime wealth. Maximize:

$$(1) \qquad E_0 \sum_{t=0}^{\infty} \beta^t u(c_t), \quad 0 < \beta < 1,$$

Subject to:

$$(2) \qquad W_{t+1} = R_{t+1}(W_t - c_t), \quad R_{t+1} = \sum_{i=0}^{n} s_t^i R_{t+1}^i, \quad \sum_{i=0}^{n} s_t^i = 1.$$

Notation is standard, with individual returns indicated by superscripts and with superscript 0 indicating the risk free asset. Portfolio shares add to 1 in each period. Returns are assumed to be uncorrelated over time. This is a key assumption that we will relax later in this chapter.

The decision problem in equations (1) and (2) can be reformulated using the dynamic programming approach (see Appendix D):

$$(3) \qquad V(W_t) = \max_{c_t,\ \{s_t^i\}_{i=1}^{n}} \left[u(c_t) + \beta E_t V(W_{t+1}) \right],$$

Subject to:

$$(4) \qquad W_{t+1} = R_{t+1}(W_t - c_t), \quad R_{t+1} = R^f_{t+1} + \sum_{i=1}^{n} s_t^i (R^i_{t+1} - R^f_{t+1}).$$

Note that the last condition is formed by combining the two last conditions in equation (2).

First-order conditions for this decision problem are for each time t :

$$(5) \qquad u_c(c_t) = \beta E_t[R_{t+1} V_W(W_{t+1})] ,$$

$$(6) \qquad E_t[(R^i_{t+1} - R^f_{t+1}) V_W(W_{t+1})] = 0, \quad \text{for all } i.$$

Subscripts denote partial derivatives except of course for the subscript t which indicates the period. Equation (5) states that the marginal utility of current consumption should be set equal to the marginal cost of decreasing real wealth by one unit, which is equal to the gross real portfolio return evaluated at how wealth affects discounted maximum expected lifetime utility at the margin. Equation (6) essentially requires that portfolios are chosen to set the expected marginal benefit of all returns equal.

The envelope condition is:

$$(7) \qquad V_W(W_t) = \beta E_t[R_{t+1} V_W(W_{t+1})].$$

Combining equations (5) and (7) produces:

$$(8) \qquad V_W(W_t) = u_c(c_t).$$

Equation (8) implies, with the help of the implicit function theorem, that current consumption is a function of current wealth only. Since the consumption function is strictly concave and since standard dynamic programming arguments show that the value function is strictly concave as well, consumption is a monotonically increasing function of wealth.

Updating equation (8) by one period and then substituting into equation (5) and separately into equation (6) yields:

$$(9) \qquad u_c(c_t) = \beta E_t[R_{t+1} u_c(c_{t+1})] ,$$

$$(10) \qquad E_t[(R^i_{t+1} - R^f_{t+1}) u_c(c_{t+1})] = 0, \quad \text{for all } i.$$

If equation (10) is multiplied by s_t^i and then summed over all i it becomes clear that the equation holds for R_{t+1} as well as for all individual returns. [Note that R_{t+1} represents the (value weighted) "market return" since we have a representative investor]. It is thus easy to show using equation (10) that equation (9) holds for all assets:

(11) $$u_c(c_t) \;=\; \beta\, E_t[R^i_{t+1}\, u_c(c_{t+1})], \quad \text{for all } i.$$

(b) The Stochastic Discount Factor

There are several implications that can be drawn from the simple model above based on the material in Chapter VII. We can write:

(12) $$E_t[m_{t+1} R^i_{t+1}] \;=\; 1 \;, \quad m_{t+1} \;=\; \beta\, u_c(c_{t+1})/u_c(c_t) \;.$$

Thus, the stochastic discount factor that prices all assets is equal to the marginal rate of intertemporal substitution. Rewards for taking risk are related only to uncertainties in aggregate consumption that cause fluctuations in marginal utility.

Similarly, we can formulate the marginal rate of intertemporal substitution in terms of wealth. Realizing that equation (7) holds for all assets *i* as well [or simply using equation (8) in equation (12)] we have:

(13) $$E_t[m_{t+1} R^i_{t+1}] \;=\; 1 \;, \quad m_{t+1} \;=\; \beta\, V_W(W_{t+1})/V_W(W_t).$$

We also know from Chapter VII that formulations (12) and (13) are equivalent to the following single-beta formulation:

(14) $$E(R^i_{t+1}) \;=\; R^i_{t+1} + \beta_{im}[E(R_{t+1}) - R^i_{t+1}] \;,$$

$$\text{where } \;\beta_{im} \;=\; Cov(R^i_{t+1}, m_{t+1})/Var(m_{t+1}) \;.$$

If m_{t+1} is given as in equation (12) we have a special case of the Consumption CAPM. A special case because of the specific assumptions made here – that returns are serially uncorrelated and that a representative consumer exists.

(c) The CAPM with Multiple Periods

We can derive the CAPM directly under the standard normality assumption. Multi-variate normality of all returns implies that end-of-period wealth is normally distributed. From equation (6), using the definition of covariance, we find:

(15) $$E_t[(R^i_{t+1} - R^f_{t+1})]\, E_t[V_W(W_{t+1})] \;=\; -Cov_t[R^i_{t+1}, V_W(W_{t+1})].$$

Employing Stein’s Lemma and using from equation (4) the fact that the covariance term can be rewritten using $R_{t+1} \;=\; W_{t+1}/(W_{t+1} - c_t)$, yields:

(16) $$E_t(R^i_{t+1}) = R^f_{t+1} + \frac{-E_t[V_{WW}(W_{t+1})]}{(W_t - c_t)E_t[V_W(W_{t+1})]} Cov_t[R^i_{t+1}, R_{t+1}].$$

The familiar next step is then to apply equation (10) to the market return:

(17) $$E_t(R_{t+1}) = R^f_{t+1} + \frac{-E_t[V_{WW}(W_{t+1})]}{(W_t - c_t)E_t[V_W(W_{t+1})]} Var_t(R_{t+1}).$$

Dividing equation (16) by (17) produces the familiar equation:

(18) $$E(R^i_{t+1}) = E(R^f_{t+1}) + \beta_i [E(R_{t+1}) - E(R^f_{t+1})] ,$$

$$\text{where} \quad \beta_i = Cov_t[R^i_{t+1}, R_{t+1}]/Var_t(R_{t+1}) .$$

Thus, the CAPM may hold in a multi-period framework as first pointed out by Fama (1970). The general requirement is for the value function to be a function of wealth only as is the case in this model. Constantinides (1980) shows that four conditions are needed for the value function to depend on wealth only: (1) homothetic utility so that current utility can be summarized in one index; (2) a representative consumer exists (as guaranteed by the assumption of complete markets); (3) utility is not state dependent; and (4) the set of returns is serially uncorrelated. The first three conditions apply to a static framework as well as in a dynamic one. We discussed the first two explicitly in Chapter IV (the sections on non-homothetic utility and intertemporal asset pricing). The three conditions are satisfied here because (1) we consider one good only, (2) we assumed a representative consumer, and (3) utility depends on the consumption good only. The fourth condition only becomes relevant in a dynamic model. This condition is relaxed in the Merton (1973) model.

(d) The Random Walk Property of Stock Prices

Returns are by definition equal to the dividend yield plus the percentage capital gain. Or:

(19) $$R^i_{t+1} = \frac{d^i_{t+1} + p^i_{t+1}}{p^i_t} ,$$

where d^i_t, p^i_t indicate, respectively, the dividend paid in period *t* on asset *i* and the price in period *t* of asset *i*. We thus consider the *ex-dividend* price of the asset. Using the definition of return in equation (11) produces:

(20) $$p^i_t u_c(c_t) = \beta E_t[(d^i_{t+1} + p^i_{t+1}) u_c(c_{t+1})],$$

for all *i* and *t*. Solving this linear first-order difference equation forward, and using the fact that the transversality condition guarantees that $\lim_{t\to\infty} \beta^t u_c(c_t) = 0$, generates:

$$\text{(21)} \qquad p_t^i = \sum_{j=1}^{\infty} \beta^j E_t[d_{t+j}^i \; u_c(c_{t+j})] / u_c(c_t) \; .$$

Asset prices equal the present value of currently expected future dividends, with the dividends each weighted at their relative marginal utility benefit at the time of receipt. Notice that under risk neutrality the marginal utility of consumption is constant. Hence the stock price of any asset *i* would be determined simply as the present value of its future dividends.

The random walk property of stock prices follows from equation (20) under an assumption of risk neutrality. For constant marginal utility, equation (20) becomes:

$$\text{(22)} \qquad E_t(d_{t+1}^i + p_{t+1}^i) = (1/\beta)\, p_t^i \; .$$

The expected value of the asset with dividends re-invested is equal to the current price plus risk-free opportunity cost. Or, in other words, the best guess of the what the value of an asset will be is the current price plus interest. Thus, the asset's value follows a random walk with drift. The share price, adjusted for discounting and dividends should be unpredictable.

(e) Bubbles

If we solve equation (22) for the current price, we get the following solution:

$$\text{(23)} \qquad p_t^i = \sum_{j=1}^{\infty} \beta^j E_t(d_{t+j}^i) + \gamma_t (1/\beta)^t \,, \qquad E_t \gamma_{t+1} = \gamma_t \;\; \text{for all } t.$$

Typically it is assumed that γ_t is zero. This solution would result directly from equation (21) for constant marginal utility. However, it is easy to check that, even for γ_t not equal to zero, equation (22) holds if we plug in equation (23) for p_t^i and p_{t+1}^i. The γ_t term is of course the "bubble" term. For a positive bubble, the price is above its fundamental as determined by future dividends. The reason that price is above fundamentals is that the bubble is assumed to keep growing: if you expect that the price will keep rising then it is rational to pay a higher current price, even if not warranted by dividend prospects. The irrational element here is that the share price is expected to grow to infinity as the bubble term increases without limit as time goes to infinity. Eventually, all the wealth in the economy is not able to purchase the share!

A popular formulation of bubbles would set $E_t \gamma_{t+1} = \gamma$ with

$$\text{(24)} \qquad \gamma_t = \begin{cases} 0 \text{ with probability F} \\ \gamma/(1-F) \text{ with probability } 1-\text{F} \; . \end{cases}$$

In this case, the bubble keeps growing exponentially until it bursts. Thus, even though everybody knows the share price is "too high" there is no reason to get out since returns are sufficiently high when the bubble does not burst.

(f) An Example for Constant Relative Risk Aversion

Assume that the preferences of the representative investor are given by:

(25) $$u(c_t) = c_t^{1-\delta} / (1-\delta).$$

To solve explicitly the savings/portfolio problem described before, assume a specific functional form for the value function:

(26) $$V(W_t) = A\, W_t^{1-\delta} / (1-\delta).$$

Now use the method of undetermined coefficients to see if a value for "A" can be found such that the budget constraint, first-order conditions and the envelope condition (or the Bellman equation itself) hold. From equations (4), (5) and (8) (the latter standing in for the envelope condition):

(27) $$W_{t+1} = R_{t+1}(W_t - c_t)$$

(28) $$c_t^{-\delta} = \beta E_t(R_{t+1} A\, W_{t+1}^{-\delta}),$$

(29) $$c_t = A^{-1/\delta} W_t.$$

Combining the above three equations generates:

(30) $$1 = \beta E_t [R_{t+1}^{1-\delta} (1 - A^{-1/\delta})^{-\delta}].$$

Thus, the conjectured form of the value function is verified if:

(31) $$A = [1 - (\beta E_t R_{t+1}^{1-\delta})^{1/\delta}]^{-\delta}.$$

By the assumption that the expected return follows a white noise process with constant distribution, *A* is indeed constant. Note that formally we should also check the Bellman equation to see if a constant should be added to the value function. It is easy to check that the constant would be zero in this case.

A special case of the example above is for $\delta = 1$. It can be shown that we obtain the logarithmic utility function in this case (formally by taking the limit of $\delta \to 1$). In this case we find that $A = 1/(1-\beta)$ in equation (31) so that $c_t = (1-\beta)W_t$ from equation (29). If we assume that the dividends of asset *i* are proportionate to aggregate consumption, i.e., $d_t^i = \omega^i c_t$, then equation (21) becomes $p_t^i = \omega^i \beta\, c_t / (1-\beta)$. The return, using equation (19) then equals: $R_{t+1}^i = (1/\beta)\, c_{t+1} / c_t$.

(g) The Random Walk Property for Consumption

Suppose that the overall return is a known constant (risk free). Then equation (9) above becomes:

$$(32) \qquad E_t[\, u_c(c_{t+1})] \;=\; (1/\beta R) u_c(c_t) \;.$$

Thus, the marginal utility of consumption follows a random walk. For quadratic preferences:

$$(33) \qquad E_t(c_{t+1}) \;=\; (1/\beta R)\, c_t \;.$$

Consumption follows a random walk. In this case it would be impossible to forecast changes in (aggregate) consumption, no matter what you try.

(h) The Timing of Consumption

The above model assumed that consumption occurs at each period in time. An alternative formulation assumes that consumption only occurs in some final period *T*. While the first approach is most common and appears to be more realistic, there are some advantages to using the second approach. This approach is more tractable and allows focus on the main impetus for saving – putting money aside for retirement. It is then easier to focus on the portfolio issues of how to change risk taking over time. As this is crucial for determining asset prices, assuming consumption in the final period only may be preferred for asset pricing models, even though it is currently uncommon to make this assumption.

In the traditional CAPM and other static asset pricing models, the timing of consumption is irrelevant since consumption occurs only once during the first and final period. Even if a formal two period model is used to derive the asset pricing implications, the first period makes no difference. Accordingly, the issue of consumption throughout or only at the end, only becomes relevant for dynamic asset pricing.

2. THE INTERTEMPORAL CAPM

(a) Merton's model with one state variable

So far we assumed that the distribution of asset returns does not change over time. In this case, even in a multi-period model, the CAPM holds under the standard assumptions (normality or ellipticality in particular). Merton (1973) based on Merton (1971) considers asset pricing under the condition that investment opportunities may change over time. He thus derives what is often called the Intertemporal CAPM (ICAPM). Merton's model is rooted in a continuous time stochastic dynamic programming framework that is technically challenging. While it is relatively easy to master a basic understanding that is sufficient to follow the literature applying the continuous time stochastic dynamic programming approach, it is quite difficult to conduct independent research using this approach. In contrast, we will derive the ICAPM in discrete time, where instead of assuming that returns follow a

Brownian motion process, or more generally a diffusion process, we assume normality.

For the sake of concreteness, we will first model a change in investment opportunities through one state variable only as in Merton (1990, Ch. 11). It is easy to generalize to the case in Merton where a vector of state variables represents changes in investment opportunities over time, which we will do in the next section. We will use the same model as in Section 1 with two modifications. First, the risk free rate changes stochastically over time. That is, while during any given period the risk free rate is known (and thus is indeed risk free), the risk free rate for the next period will change stochastically during the current period. This is the simplest way to change the set of investment opportunities over time. Returns on all other securities are assumed to be uncorrelated over time and are multi-variate normally distributed conditional on the risk free rate for the upcoming period. Second, we no longer assume that a representative consumer exists and so consider the decision problem for some investor *k*.

$$(1) \qquad V^k(W_t^k, R_t^f) = \max_{c_t^k,\ \{s_t^{ik}\}_{i=1}^n} [u^k(c_t^k) + \beta_k E_t V^k(W_{t+1}^k, R_{t+1}^f)] ,$$

Subject to:

$$(2) \qquad W_{t+1}^k = R_{t+1}^k (W_{t+1}^k - c_t^k) , \quad R_{t+1}^k = R_t^f + \sum_{i=1}^n s_t^{ik} (R_{t+1}^i - R_t^f) .$$

$$(3) \qquad R_{t+1}^f = F(R_t^f, \epsilon_{t+1})$$

Note that the risk free rate pre-set in period *t* is relevant for the portfolio return in period *t+1*. The value function now depends on the current risk free rate as an additional state variable. Clearly, wealth has a different meaning when risk free rates are high compared to when they are low.

First-order conditions for this decision problem are for each time *t* :

$$(4) \qquad u_c^k(c_t^k) = \beta_k E_t [R_{t+1}^k V_W^k (W_{t+1}^k, R_{t+1}^f)] ,$$

$$(5) \qquad E_t [(R_{t+1}^i - R_t^f) V_W^k (W_{t+1}^k, R_{t+1}^f)] = 0, \quad \text{for all } i.$$

We will now use equation (5) to derive a beta asset pricing equation. Using the definition of covariance and converting to net returns:

$$(6) \qquad E_t [V_W^k (W_{t+1}^k, r_{t+1}^f)] (\mu_{t+1}^i - r_t^f) = -Cov_t [V_W^k (W_{t+1}^k, r_{t+1}^f), r_{t+1}^i].$$

Given the assumption that returns are normally distributed so that w_k is normal and that *p* is normally distributed, we can again apply my modest *generalization of Stein's Lemma* [see Appendix C] stating that, when *x, y,* and *z* are multivariate normal, then:

$$\text{Cov}[x, h(y,z)] = E[h_1(y,z)]\text{Cov}(x,y) + E[h_2(y,z)]\text{Cov}(x,z) .$$

Thus, applying the lemma to equation (6):

$$\text{(7)} \qquad \mu^i_{t+1} - r^f_t = \frac{-E_t[V^k_{WW}(W^k_{t+1}, r^f_{t+1})]}{E_t[V^k_W(W^k_{t+1}, r^f_{t+1})]} \, Cov_t(W^k_{t+1}, r^i_{t+1})$$

$$+ \frac{-E_t[V_{Wr^f}(W^k_{t+1}, r^f_{t+1})]}{E_t[V^k_W(W^k_{t+1}, r^f_{t+1})]} \, Cov_t(r^f_{t+1}, r^i_{t+1}).$$

Equation (7) suggests a two-factor CAPM result. However, the expression includes various terms that are specific to individual *k*. The next step thus is to consider market equilibrium by aggregating over all individuals:

$$\text{(8)} \qquad \sum_{k=1}^{K} \frac{E_t[V^k_W(W^k_{t+1}, r^f_{t+1})]}{-E_t[V^k_{WW}(W^k_{t+1}, r^f_{t+1})]} (\mu^i_{t+1} - r^f_t) = Cov_t(\sum_{k=1}^{K} W^k_{t+1}, r^i_{t+1})$$

$$+ \sum_{k=1}^{K} \frac{E_t[V^k_{Wr^f}(W^k_{t+1}, r^f_{t+1})]}{E_t[V^k_{WW}(W^k_{t+1}, r^f_{t+1})]} \, Cov_t(r^f_{t+1}, r^i_{t+1}) .$$

The first covariance term can be rewritten using the fact that aggregate wealth in period *t+1* must be equal to $R^m_{t+1} W^m_t$. That is the gross market return times initial market wealth. Note though that initial market wealth should be investor wealth net of aggregate consumption and this should in turn be equal to the value of the aggregate asset portfolio.

Thus we can write:

$$\text{(9)} \qquad \mu^i_{t+1} - r^f_t = A_t \, Cov_t(r^m_{t+1}, r^i_{t+1}) + B_t \, Cov_t(r^f_{t+1}, r^i_{t+1}),$$

where:

$$A_t = W^m_t \Big/ \sum_{k=1}^{K} \frac{E_t[V^k_W(W^k_{t+1}, r^f_{t+1})]}{-E_t[V^k_{WW}(W^k_{t+1}, r^f_{t+1})]} ,$$

$$B_t = \sum_{k=1}^{K} \frac{E_t[V^k_{Wr^f}(W^k_{t+1}, r^f_{t+1})]}{E_t[V^k_{WW}(W^k_{t+1}, r^f_{t+1})]} \Big/ \sum_{k=1}^{K} \frac{E_t[V^k_W(W^k_{t+1}, r^f_{t+1})]}{-E_t[V^k_{WW}(W^k_{t+1}, r^f_{t+1})]} .$$

Applying equation (9) to asset *m* (it is easy to check that, if equation (5) holds for any "primitive" asset *i* , it also hold for any portfolio, including the market portfolio):

$$(10)\qquad \mu^m_{t+1} - r^f_t \;=\; A_t\, Var_t(r^m_{t+1}) \;+\; B_t\, Cov_t(r^f_{t+1}\,,\, r^m_{t+1}).$$

Similarly, for an asset with return perfectly correlated with *next period*'s risk free return,

$$(11)\qquad \mu^f_{t+1} - r^f_t \;=\; A_t\, Cov_t(r^m_{t+1}\,,\, r^f_{t+1}) \;+\; B_t\, Var_t(r^f_{t+1})\;.$$

Use equations (10) and (11) to solve for A_t and B_t :

$$\begin{pmatrix} A_t \\ B_t \end{pmatrix} \;=\; \begin{pmatrix} Var_t(r^m_{t+1}) & Cov_t(r^f_{t+1}\,,\, r^m_{t+1}) \\ Cov_t(r^m_{t+1}\,,\, r^f_{t+1}) & Var_t(r^f_{t+1}) \end{pmatrix}^{-1} \begin{pmatrix} \mu^m_{t+1} - r^f_t \\ \mu^f_{t+1} - r^f_t \end{pmatrix}.$$

Substitute the solution for A_t and B_t into equation (9) to obtain the expected return of any asset *i* as:

$$(12)\qquad \mu^i_{t+1} - r^f_t \;=\; \beta_{im}\,(\mu^m_{t+1} - r^f_t) \;+\; \beta_{if}\,(\mu^f_{t+1} - r^f_t)\;,$$

with:

$$\beta_{im} \;=\; \frac{Var_t(r^f_{t+1})\, Cov_t(r^i_{t+1}\,,\, r^m_{t+1}) - Cov_t(r^f_{t+1}\,,\, r^m_{t+1})\, Cov_t(r^f_{t+1}\,,\, r^i_{t+1})}{Var_t(r^f_{t+1})\, Var_t(r^m_{t+1}) - [Cov_t(r^f_{t+1}\,,\, r^m_{t+1})]^2}\;,$$

$$\beta_{if} \;=\; \frac{Var_t(r^m_{t+1})\, Cov_t(r^i_{t+1}\,,\, r^f_{t+1}) - Cov_t(r^f_{t+1}\,,\, r^m_{t+1})\, Cov_t(r^m_{t+1}\,,\, r^i_{t+1})}{Var_t(r^f_{t+1})\, Var_t(r^m_{t+1}) - [Cov_t(r^f_{t+1}\,,\, r^m_{t+1})]^2}$$

Notice that the betas are simply the slope coefficients that would arise in a multi-variate regression, given that the conditional covariances are constant over time.

The intuition of equation (12) is that an investor is faced with two types of systematic risk. First the risk of wealth fluctuation; second, changes in investment opportunities related to changes in the risk free rate. If the risk free rate falls then, for an investor with positive net wealth, this is tantamount to a decrease in future consumption opportunities. The investor would like to hedge herself against such a decrease in future risk free rates by keeping securities in her portfolio the returns of which are negatively correlated with future interest rates. Thus, the risk premium, $\mu^f_{t+1} - r^f_t$, on the asset that is perfectly correlated with the future risk free rate (that is, the second factor) should be positive. Note that this observation explains the fact that the yield curve, all else equal, should have a positive slope. Empirically, one could take the return on long-term government bonds to represent the second factor. The "term premium" in fact has been shown in *ad hoc* empirical work to have positive predictive value for future returns.

The original Merton model includes multiple factors that represent the current state of the investment opportunity set. In particular, another factor could be the equity premium or any other factor affecting future investment returns. The Merton model does not specify what these factors could be. In that sense the Merton model is very similar to the APT in that it provides a license for a fishing expedition. Just about any variable that is significant in explaining the cross-section of asset returns can be motivated as somehow affecting future investment opportunities. Either directly or as a proxy for a variable that directly affects future investment opportunities. It is a relatively trivial matter to conform the above model more to the Merton model by allowing an arbitrary number of factors that affect future investment opportunities, as we show in the next sub-section.

For the sake of testability, it may be useful to attempt to derive specific factors that impact future investment opportunities. A good starting point would be to construct a general equilibrium model of the factors affecting the equity premium over time. (The next chapter will discuss general equilibrium models of this type). These factors should then be added to the "yield curve" factor discussed here. Another issue though is that these factors will need to be clearly identifiable empirically.

A final issue is of a more technical nature. The maintained assumption was that the future risk free rate is conditionally normally distributed. This assumption is not essential. To apply Stein's Lemma all that is needed is that some underlying shock is normally distributed. Thus, we may just assume that the shock ϵ_{t+1} in equation (3) in normally distributed. If we make this assumption then results will basically be unchanged. In equation (9), B_t will become:

$$(13) \qquad B_t = \sum_{k=1}^{K} \frac{E_t[V^k_{Wr^f}(W^k_{t+1}, r^f_{t+1})F_\epsilon(r^f_t, \epsilon_{t+1})]}{E_t[V^k_{WW}(W^k_{t+1}, r^f_{t+1})]\, E[F_\epsilon(r^f_t, \epsilon_{t+1})]} \Big/ \sum_{k=1}^{K} \frac{E_t[V^k_W(W^k_{t+1}, r^f_{t+1})]}{-E_t[V^k_{WW}(W^k_{t+1}, r^f_{t+1})]} .$$

This follows by applying Stein's lemma now using ϵ_{t+1} as a basic variable; and subsequently by applying Stein's lemma in reverse to return to r^f_{t+1} as the basic variable. No other changes are necessary. Clearly, r^f_{t+1} need not be normally distributed. For instance if *F( )* is an exponential function then r^f_{t+1} is log-normal. Note that this "trick" can be applied generally. The outcome thus closely parallels the situation in continuous time formulations. There the building blocks must be Brownian motion which is the continuous time equivalent of a normally distributed process.

(b) Merton's model with s *state variables*

Here we derive the general version of Merton's model.

$$(14) \qquad V^k(W^k_t, \boldsymbol{h_t}) = \max_{c^k_t,\ \boldsymbol{s^k_t}} \left[u^k(c^k_t) + \beta_k E_t V^k(W^k_{t+1}, \boldsymbol{h_{t+1}}) \right] ,$$

Subject to:

$$(15) \qquad W^k_{t+1} = R^k_{t+1}(W^k_{t+1} - c^k_t), \quad R^k_{t+1} = R^f_{t+1}(\boldsymbol{h_t}) + (\boldsymbol{s^k_t})' [\boldsymbol{R_{t+1}}(\boldsymbol{h_t}, \boldsymbol{\varepsilon_{t+1}}) - R^f_{t+1}(\boldsymbol{h_t})].$$

$$(16) \qquad \boldsymbol{h_{t+1}} = \boldsymbol{F}(\boldsymbol{h_t}, \boldsymbol{\eta_{t+1}}),$$

with notation similar as in the previous section except that $\boldsymbol{h_t}$ represents the vector of all state variables that affect investment opportunities; that is the returns on all assets as indicated in equation (2).

First-order conditions for this decision problem are for each time t :

$$(17) \qquad u_c^k(c_t^k) = \beta_k E_t[R_{t+1}^k V_W^k(W_{t+1}^k, \boldsymbol{h}_{t+1})] \; ,$$

$$(18) \qquad E_t[(\boldsymbol{R}_{t+1} - R_{t+1}^f \mathbf{1}) V_W^k(W_{t+1}^k, \boldsymbol{h}_{t+1})] = \mathbf{0},$$

Using the definition of covariance and converting to net returns:

$$(19) \qquad E_t[V_W^k(W_{t+1}^k, \boldsymbol{h}_{t+1})](\boldsymbol{\mu}_{t+1} - r_{t+1}^f \mathbf{1}) = -Cov_t[V_W^k(W_{t+1}^k, \boldsymbol{h}_{t+1}), \boldsymbol{r}_{t+1}].$$

Applying Stein's Lemma:

$$(20) \qquad \boldsymbol{\mu}_{t+1} - r_{t+1}^f \mathbf{1} = \frac{-E_t[V_{WW}^k(W_{t+1}^k, \boldsymbol{h}_{t+1})]}{E_t[V_W^k(W_{t+1}^k, \boldsymbol{h}_{t+1})]} Cov_t(W_{t+1}^k, \boldsymbol{r}_{t+1}) + \frac{-\boldsymbol{E_t[V_{Wh}^k}(W_{t+1}^k, \boldsymbol{h}_{t+1})]'}{E_t[V_W^k(W_{t+1}^k, \boldsymbol{h}_{t+1})]} \boldsymbol{Cov_t}(\boldsymbol{h}_{t+1}, \boldsymbol{r}_{t+1}).$$

Aggregating over all individuals k yields:

$$(21) \qquad \sum_{k=1}^{K} \frac{E_t[V_W^k(W_{t+1}^k, \boldsymbol{h}_{t+1})]}{-E_t[V_{WW}^k(W_{t+1}^k, \boldsymbol{h}_{t+1})]} (\boldsymbol{\mu}_{t+1} - r_t^f) = Cov_t(\sum_{k=1}^{K} W_{t+1}^k, \boldsymbol{r}_{t+1}) + \sum_{k=1}^{K} \frac{-\boldsymbol{E_t[V_{Wh}^k}(W_{t+1}^k, \boldsymbol{h}_{t+1})]'}{E_t[V_{WW}^k(W_{t+1}^k, \boldsymbol{h}_{t+1})]} \boldsymbol{Cov_t}(\boldsymbol{h}_{t+1}, \boldsymbol{r}_{t+1}) \; .$$

Thus,

$$(22) \qquad \boldsymbol{\mu}_{t+1} - r_{t+1}^f \mathbf{1} = a_t \, Cov_t(r_{t+1}^m, \boldsymbol{r}_{t+1}) + \boldsymbol{b}_t' \, Cov_t(\boldsymbol{h}_{t+1}, \boldsymbol{r}_{t+1}),$$

where:

$$a_t = W_t^m \Big/ \sum_{k=1}^{K} \frac{E_t[V_W^k(W_{t+1}^k, \boldsymbol{h}_{t+1})]}{-E_t[V_{WW}^k(W_{t+1}^k, \boldsymbol{h}_{t+1})]} \; ,$$

$$\boldsymbol{b_t} = \sum_{k=1}^{K} \frac{\boldsymbol{E_t}[V_{W\boldsymbol{h}}^k(W_{t+1}^k, \boldsymbol{h_{t+1}})]}{E_t[V_{WW}^k(W_{t+1}^k, \boldsymbol{h_{t+1}})]} \Big/ \sum_{k=1}^{K} \frac{E_t[V_W^k(W_{t+1}^k, \boldsymbol{h_{t+1}})]}{-E_t[V_{WW}^k(W_{t+1}^k, \boldsymbol{h_{t+1}})]} \ .$$

One particular element of the set of all assets could be the market asset; thus:

$$(23) \qquad \mu_{t+1}^m - r_{t+1}^f = a_t \, Var_t(r_{t+1}^m) + \boldsymbol{b_t}' \, Cov_t(r_{t+1}^m, \boldsymbol{h_{t+1}}).$$

Similarly, for any asset with return perfectly correlated with any of the state variables,

$$(24) \qquad \boldsymbol{\mu_{t+1}^h} - r_{t+1}^f \mathbf{1} = a_t \, Cov_t(r_{t+1}^m, \boldsymbol{h_{t+1}}) + \boldsymbol{b_t}' \, Var_t(\boldsymbol{h_{t+1}}).$$

Use the above two equations to solve for a_t and $\boldsymbol{b}_t$:

$$(25) \qquad \begin{pmatrix} a_t \\ \boldsymbol{b}_t \end{pmatrix} = \begin{pmatrix} Var_t(r_{t+1}^m) & \boldsymbol{Cov_t(h_{t+1}, r_{t+1}^m)} \\ \boldsymbol{Cov_t(r_{t+1}^m, h_{t+1})} & \boldsymbol{Cov_t(h_{t+1})} \end{pmatrix}^{-1} \begin{pmatrix} \mu_{t+1}^m - r_{t+1}^f \\ \boldsymbol{\mu_{t+1}^h} - r_{t+1}^f \mathbf{1} \end{pmatrix}.$$

$$(26) \qquad \boldsymbol{\mu_{t+1}} - r_{t+1}^f \mathbf{1} = \begin{pmatrix} Cov_t(r_{t+1}^m, \boldsymbol{r_{t+1}}) \\ Cov_t(\boldsymbol{h_{t+1}}, \boldsymbol{r_{t+1}}) \end{pmatrix}' \begin{pmatrix} Var_t(r_{t+1}^m) & \boldsymbol{Cov_t(h_{t+1}, r_{t+1}^m)} \\ \boldsymbol{Cov_t(r_{t+1}^m, h_{t+1})} & \boldsymbol{Cov_t(h_{t+1})} \end{pmatrix}^{-1} \begin{pmatrix} \mu_{t+1}^m - r_{t+1}^f \\ \boldsymbol{\mu_{t+1}^h} - r_{t+1}^f \mathbf{1} \end{pmatrix}.$$

Equation (26) implies a linear equation with the coefficients given as those in a multiple regression.

3. THE CONSUMPTION CAPM

We know from the material in Chapter VI that there exists a single-beta formulation for every asset pricing model (if arbitrage opportunities are ruled out). Breeden (1979) first discovered this formulation for the ICAPM by taking advantage of the envelope condition in the stochastic dynamic programming framework. Employing the same model as in the previous section, consider the implication of using the envelope condition and rewriting the first-order conditions. This yields the equivalent of equation (I.8) in the model with changing investment opportunities and without a representative investor:

$$(1) \qquad V_W^k(W_t^k, r_t^f) = u_c^k(c_t^k) \ .$$

Wealth has value at the margin since it allows additional consumption at the margin (consumption smoothing implies that just current marginal utility of consumption is sufficient to capture the marginal benefits of additional wealth). As equation (1) shows, one may think of the marginal utility of consumption as a sufficient statistic of the separate variables that affect the marginal effect of wealth on lifetime utility.

Substituting equation (1) into equation (II.6) yields:

$$(2) \qquad E_t[u_c^k(c_{t+1}^k)](\mu_{t+1}^i - r_t^f) = -Cov_t[u_c^k(c_{t+1}^k), r_{t+1}^i].$$

Assume that consumption of each investor is (conditionally) normally distributed. Then applying Stein's Lemma to equation (2) produces:

$$(3) \qquad -E_t[u_c^k(c_{t+1}^k)](\mu_{t+1}^i - r_t^f) = E[u_{cc}^k(c_{t+1}^k)]\, Cov_t[c_{t+1}^k, r_{t+1}^i] .$$

Summing over all investors after dividing by the term in front of the covariance gives:

$$(4) \qquad (\mu_{t+1}^i - r_t^f) \sum_{k=1}^{K} \frac{-E_t[u_c^k(c_{t+1}^k)]}{E[u_{cc}^k(c_{t+1}^k)]} = Cov_t[c_{t+1}, r_{t+1}^i] .$$

Note that the covariance term on the right-hand side now has aggregate consumption as an argument, which is clearly the sum of consumption of all individual investors added together.

If an asset exists with return that is conditionally perfectly correlated with consumption such that $c_{t+1} = g_t + h_t r_{t+1}^c$, then

$$(5) \qquad \mu_{t+1}^i - r_t^f = H_t\, Cov_t(r_{t+1}^c, r_{t+1}^i) ,$$

with:

$$H_t = h_t / \sum_{k=1}^{K} \frac{-E_t[u_c^k(c_{t+1}^k)]}{E[u_{cc}^k(c_{t+1}^k)]} .$$

Applying equation (5) to the asset with return perfectly correlated with consumption:

$$(6) \qquad \mu_{t+1}^c - r_t^f = H_t\, Var_t(r_{t+1}^c) .$$

Combining equations (5) and (6) then generates the Consumption-Based CAPM (or CCAPM):

$$(7) \qquad \mu_{t+1}^i - r_t^f = \beta_{ic}\,(\mu_{t+1}^c - r_t^f), \text{ with}$$

$$\beta_{ic} = Cov_t(r_{t+1}^c, r_{t+1}^i) / Var_t(r_{t+1}^c) .$$

In empirical applications the return on the asset perfectly correlated with consumption is usually taken to be the growth rate of consumption which is more easily observable. However, the CCAPM operationalized in this way does not perform well. One reason might be the fact that observed consumption is not similar enough to the theoretical concept of consumption. It is difficult for instance to measure the stream of durable good consumption services during one period. It certainly is not equal to the amount currently spent on consumer durables. The empirical work of Mankiw and Shapiro (1986) calculates both a market beta and a consumption beta for each of 464 stocks from 1959-1982. When both are included in a cross-sectional regression, the market beta clearly outperforms the consumption beta in explaining the cross-sectional variation in returns.

If no asset exists that is closely enough correlated with consumption, equation (6) cannot be used. Instead, define $r^g_{t+1} = c_{t+1} / c_t$ as the growth rate of aggregate consumption. Then equation (5) becomes:

$$(8) \qquad \mu^i_{t+1} - r^f_t = K_t \, Cov_t(r^g_{t+1}, r^i_{t+1}) \;,$$

with:

$$K_t = c_t / \sum_{k=1}^{K} \frac{-E_t[u^k_c(c^k_{t+1})]}{E[u^k_{cc}(c^k_{t+1})]} \;.$$

Consider now an asset that is highly correlated with consumption. For instance, the market:

$$(9) \qquad \mu^m_{t+1} - r^f_t = K_t \, Cov_t(r^g_{t+1}, r^m_{t+1}).$$

Then we find the asset pricing equation:

$$(10) \qquad \mu^i_{t+1} - r^f_t = \beta_{igm} \, (\mu^m_{t+1} - r^f_t) \;, \quad \text{where:}$$

$$\beta_{igm} = Cov_t(r^g_{t+1}, r^i_{t+1}) / Cov_t(r^g_{t+1}, r^m_{t+1}).$$

Empirically, this version of the CCAPM is estimated using an IV technique: first regress the growth rate of consumption on the market return; then use the predicted value of the growth rate of consumption based on the market return to regress against the time series of returns of each asset. This yields the appropriate beta.

4. STYLIZED FACTS AND PUZZLES IN DYNAMIC ASSET PRICING

Here we consider some empirical regularities that are important considerations in dynamic asset pricing models.

(a) The Equity Premium Puzzle

Historically, market returns (the return on a comprehensive U.S. stock market index, such as the CRSP value-weighted index) have exceeded the riskless rate by around 7%. This excess return is much larger than can be explained by the standard model leading to the CCAPM as described for instance in equation (9). The standard assumption is that of CRRA. In this case, the only way that the data can be matched with equation (9) is if the degree of CRRA far exceeds 10. This conflicts greatly with typical estimates of the degree of CRRA that vary between 0.5 and 2.0. Additionally, a CRRA of 10 $[u_c(c_t) = c_t^{-10}]$ implies such an extreme aversion to gambles that introspection reveals the unreasonability of this number. In particular, given an initial consumption level of $10,000, for a 50% chance of winning $1000, an investor with these preferences would pay only $374 (make sure to check!).

An associated puzzle is the fact that the real risk free rate is too low; around 1%. Typical asset pricing models again have a hard time explaining this fact. The equity premium puzzle was first proposed by Mehra and Prescott (1982). Typically, proposed solutions related to liquidity constraints, habit persistence, etc., can explain the equity premium puzzle, but then cannot explain the risk-free rate puzzle. A good survey of the current success, or lack thereof, in explaining the puzzle is found in Kocherlakota (1996). One explanation is that the equity premium is, in fact, much lower than 7%; Fama and French (2001) estimate an equity premium as low as around 3%.

(b) The Excess Volatility Puzzle

It is easy to feel that stock prices vary much more than is warranted by movements in fundamentals (factors that determine future dividends and factors that determine the rate at which future dividends are discounted). Shiller (1981) formalized this concern. A common explanation for high volatility is that prices just react to new information. Clearly, when the information turns out to be incorrect the stock price will return to its previous value. This would be an event that can be explained from rational behavior. However, Shiller showed that rational learning requires that the reaction to new information should be damped: the existing information should have some weight if the new information is not 100% reliable. Based on this reasoning, Shiller finds that the volatility of stock prices should be less than the volatility of the fundamentals. He takes realized dividends as the fundamentals. Historical data then show that volatility of stock prices is four times as high as the volatility of the fundamental price. This clearly conflicts with the rational-behavior-based prediction that the volatility of stock prices should be less than the volatility of the fundamental price.

(c) Predictability of Returns

Fama and French (1988) and Poterba and Summers (1989) have shown that over longer horizons (one to five years) stock *prices* display *mean reversion*. Trend reversion is probably a better descriptor. When asset prices deviate from trend they have a slow tendency to revert back to trend. A similar phenomenon is that stock *returns* exhibit negative serial correlation: a high positive excess return will be offset slowly by negative excess returns as the price slowly reverts to trend. Due to the fact that mean reversion is slow, it can only be picked up accurately over longer horizons. Unfortunately, reliable long time series of stock returns, extending prior to 1926, are hard to come by. As a result, the empirical tests for mean reversion lack power, and the evidence of mean reversion is controversial.

DeBondt and Thaler (1985) had previously uncovered a related phenomenon: profitability of "contrarian" strategies. Investing in those stocks that have performed the worst over the previous three to five years and shorting those stocks that have performed the best during that period produces significant positive excess returns.

In contrast, Lo and MacKinlay (1988) and Jegadeesh and Titman (1993) find positive autocorrelation – "momentum" – at short horizons. Over the course of a few days up to a half year or so, positive (negative) excess returns tend to be followed by further positive (negative) excess returns.

The empirical regularities discussed here – equity premium, excess volatility, and predictability – may have rational explanations. However, it is tempting to explain all three from a non-rational perspective: individuals overreact to risk; they also overreact to new information; as prices overreact to information they must eventually come down to fundamentals, which implies mean reversion.